PROPERTIES OF FIXED POINT SPACES

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In this note relations between the fixed point property and compactness are studied and it is shown that the fixed point property need not be preserved under the cross product. Most known theorems concerning the fixed point property (written as the "f.p.p.") are for compact spaces. It is to be shown that, in the general case, compactness and the f.p.p. are only vaguely related. Example 1 is an example of a Hausdorff space that has no compact subsets except finite sets. By the use of Theorem 1, it will be shown that this space has the f.p.p.

Theorem 2 states a weak compactness condition that metric spaces with the f.p.p. must possess—namely that every infinite chain of arcs must have a nonvoid limiting set. The space in Example 2 has the f.p.p., yet its cross product with itself does not satisfy this arc-compactness condition, and thus cannot have the f.p.p.

Example 3 is an example of a locally contractible metric space which has the f.p.p. yet is not compact. Theorem 3 states that if X is a locally connected, locally compact metric space with the f.p.p., then X is compact.

Example 4 is an example of a compact metric space X which does not have the f.p.p., yet contains a dense subset Y which does have the f.p.p. Thus the f.p.p. is not preserved under closure.

THEOREM 1. *Suppose X is a set and v is a topology for X such that (X, v) is a regular space with the f.p.p. If u is a stronger topology for X (i.e., contains more open sets) such that if R is open in u, its closure is the same in both topologies, then (X, u) has the f.p.p.*

PROOF. Suppose that f is a function from X into X that is continuous in (X, u). It will be shown that f is continuous in (X, v) and thus must have a fixed point.

Suppose $p \in X$ and O' is any open set of v containing $f(p)$. Since (X, v) is regular, there exists a set O that is open in v, contains $f(p)$ and $\overline{O} \subset O'$. The closure of O is the same in both topologies, since O is also open in (X, u). $D = f^{-1}(O)$ is in u because O is an open set of (X, u) and f is continuous in (X, u). $\overline{D}$ is the same in both topologies and is closed in each. Therefore, $D^{-\sim}$, the complement of $\overline{D}$, is open in each. $D^{-\sim-}$ is the same in both topologies, is closed in each, and contains $D^{-\sim}$. Therefore $D^{-\sim-\sim}$ is open in both spaces, is contained in $\overline{D}$, and contains p. Now $\overline{D} = [f^{-1}(O)]^- \subset f^{-1}(\overline{O}) \subset f^{-1}(O')$. Thus

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there is an open set of u which contains p and whose image under f is contained in O'. Therefore, f is continuous in (X, v) and thus must have a fixed point.

EXAMPLE 1. This is an example of a Hausdorff space which has the f.p.p. yet contains no compact subsets except for finite sets.

Let X be the unit interval and u be the collection of all subsets S such that there exists an open set A and a countable (infinite or finite) set B so that $S=A-B$. It is clear that the intersection of any two such sets is such a set. Also it can be shown that an arbitrary union of sets of this type is again a set of this type, and thus that u is a topology for X. In order to show this, note that the union of any collection of sets of this type is equal to the union of a countable subcollection except for a countable number of points.

Let v be the ordinary topology for X. It must be shown that $\bar{S}^v=\bar{S}^u$. Since u is stronger than v, it must always be true that $\bar{S}^v \supset \bar{S}^u$. To show inclusion the other way, suppose p is a limit point of S in (X, v) and $S'=A-B$ contains p, where A is an ordinary open set and B is countable. Since A must intersect S at uncountably many points, $A-B$ must intersect S. Therefore p is also a limit point of S in (X, u). Thus $\bar{S}^v=\bar{S}^u$.

Now according to Theorem 1, the unit interval must have the f.p.p. under this new topology. No countable set can have a limit point under this topology, therefore the space contains no infinite compact subsets. Thus there does not exist a point so that the space is locally compact at that point. This space is, however, pseudocompact. A topological space is pseudocompact if every continuous real-valued function is bounded. It can be shown that if $\{O_n\}$ is a countable open covering of $(\bar{X}, u)$, then $\{\bar{O}_n\}$ has a finite subcovering. This implies pseudocompactness (see, e.g., [1]).

The preceding example does not satisfy the first axiom of countability and thus is not metrizable. If a fixed point space is metric, must it possess some type of compactness? This question is answered in part by Theorem 2, which states that in a metric space with the f.p.p., every infinite chain of arcs must have a nonvoid limiting set. In a space that contains "enough" arcs, this is a type of compactness. First, it will be necessary to prove 2 lemmas.

DEFINITION. An arc is a homeomorph of the unit interval.

DEFINITION. A countable set of arcs $\{A_n\}=\{[b_n, c_n]\}$ is a chain if $c_n=b_{n+1}$ for all n.

DEFINITION. A collection of sets $\{S_n\}$ is locally finite if for each point p of the space, there exists an open set containing p and intersecting only finitely many S_n.

LEMMA. *If X is a connected, locally connected metric space, M a compact subset of X, and D an open set with $M \subset D$ and $\bar{D}$ compact, then only a finite number of the components of $\tilde{M}$ intersect $D^{-\sim}$.*

PROOF. Suppose $\{E_i\}$ is an infinite collection of distinct components of $\tilde{M}$, each intersecting $D^{-\sim}$. Since X is locally connected and $\tilde{M}$ is open, each E_i is a connected open set. Since the space is connected, each E_i has a limit point in M, and thus intersects D as well as $D^{-\sim}$. Each E_i is connected and thus intersects $\bar{D}-D$ at some point, say p_i. Since $\bar{D}-D$ is closed and compact and the p_i's are distinct points, they have a limit point p of $\bar{D}-D$. Now p is in some open component E of $\tilde{M}$, yet E can be no more than one of the E_i's and thus can intersect no more than one E_i. Thus at most one p_i is in E, and p cannot be a limit point of the p_i's. This is a contradiction and proves the lemma.

LEMMA. *If X is a connected, locally connected, locally compact metric space that is noncompact, then its one point compactification is a Peano continuum.*

PROOF. Let $(X+p)$ be the one point compactification of X. $(X+p)$ is connected and compact. The space X is separable [**5**, p. 75]. Therefore it has a countable open covering $\{O_i\}$ with each $\bar{O}_i$ compact. Let $M_n = \bigcup_{i=1}^{n} \bar{O}_i$ and note that $\{\tilde{M}_n + p\}$ is a countable base at p in $(X+p)$. Thus the space $(X+p)$ has a countable base, and since it is regular [**3**, p. 150] it must be metrizable [**3**, p. 125].

The connected compact metric space $(X+p)$ will be a Peano continuum if it is locally connected at p.

Suppose O is an open set of $(X+p)$ containing p. Then $M=\tilde{O}$ is compact in X. Since X is locally compact and M is compact, there exists an open set D (of X) with $D \supset M$ and $\bar{D}$ compact in X. Applying the previous lemma, only a finite number of components of $\tilde{M}$ intersection $D^{-\sim}$. Let R denote the union of those components of $\tilde{M}$ intersecting $D^{-\sim}$ which have noncompact closure. R is nonvoid since X is noncompact. Now $(R+p)$ is connected and open in $(X+p)$ and contained in O. Thus $(X+p)$ is locally connected at p and thus locally connected.

THEOREM 2. *If X is a metric space with the f.p.p., then every locally finite chain of arcs is finite.*

PROOF. Suppose $\{An\}$ is a locally finite infinite chain of arcs in X. Denote their union by A. A is a connected, locally connected, locally compact metric space in the relative topology. Its one point compactification $(A+p)$ is a Peano continuum by the previous lemma. Since a Peano space is arcwise connected [**6**, p. 81], $(A+p)$ contains an arc $[a, p]$ with p as one end point. From Borsuk's result [**2**, Corollary 14] it follows that the space X cannot have the f.p.p. The idea used in the proof of the corollary is to show that the "half line" $[a, p)$, which is closed in X, is a retract of X, and thereby show X cannot have the f.p.p.

Example 2. This is an example of a metric space X that has the f.p.p. and yet the cross product of X with itself does not have the f.p.p.

Let $f(x)=\sin \pi/(1-x)$ if $0 \leqq x < 1$ and $f(x)=1$ if $x=1$. X is the set of points $(x, f(x))$ of E_2 with $0 \leqq x \leqq 1$. The topology for X is the relative topology. Suppose g is a function from X to X with no fixed point. Let S_1 be the set of points whose x-coordinate moves to the right and S_2 be the set of all points whose x-coordinate moves to the left. $S_1 \cup S_2 = [0, 1]$ and thus one of the sets must contain a limit point p of the other. The point $(p, f(p))$ must be a fixed point.

Now it will be shown that the space $X \times X$ does not have the f.p.p. This cross product space has a natural embedding in E_4 as follows: $X \times X$ is all $(a, f(a), b, f(b))$ for $0 \leqq a \leqq 1$ and $0 \leqq b \leqq 1$ with the relative topology. Now define a subspace C of E_3 as follows: C is all $(x, f(x)+f(z), z)$ for $0 \leqq x \leqq 1$ and $0 \leqq z \leqq 1$. C and $X \times X$ are homeomorphic. Consider the one-to-one function $h(a, f(a), b, f(b)) = (a, f(a)+f(b), b)$ from $X \times X$ onto C. Suppose $(a_n, f(a_n), b_n, f(b_n))$ is a sequence approaching $(a, f(a), b, f(b))$ in $X \times X$. Then $a_n \to a$, $f(a_n) \to f(a)$, $b_n \to b$, and $f(b_n) \to f(b)$. Then $(a_n, f(a_n)+f(b_n), b_n) \to (a, f(a)+f(b), b)$ in C. Thus h is continuous. Now it will be shown that h^{-1} is continuous. Suppose $(a_n, f(a_n)+f(b_n), b_n) \to (a, f(a)+f(b), b)$ in C. Then $a_n \to a$, $b_n \to b$, and $f(a_n)+f(b_n) \to f(a)+f(b)$. If a is a point of continuity of f, i.e., if $a \neq 1$, then $f(a_n) \to f(a)$ and so $f(b_n) \to f(b)$. A similar argument holds if $b \neq 1$. Now suppose $a=b=1$. In this case, $f(a)+f(b)=2$. Since $f(a_n) \leqq 1$ and $f(b_n) \leqq 1$, it follows that $f(a_n) \to 1 = f(a)$ and $f(b_n) \to 1 = f(b)$. Thus in any of the above cases, $(a_n, f(a_n), b_n, f(b_n)) \to (a, f(a), b, f(b))$ in $X \times X$. Thus h^{-1} is continuous and h is a homeomorphism.

Now an infinite locally finite chain of arcs will be constructed in C, and thus, by Theorem 2, C will not have the f.p.p. Note that C is all (x, y, z) where x and z are in $[0, 1]$ and $y = f(x)+f(z)$. Roughly, the construction will be as follows. The first arc will simply be one hump of the sine wave in the xy plane. It will start at the origin and follow the sine wave in the xy plane to the first zero. The second arc will start there and follow a sine curve parallel to the yz plane to the next zero. The third arc will start there and be another hump of a sine wave in a plane parallel to the xy plane. In this manner the arcs will "approach" in a zig-zag manner the vertical line $\{(x, y, z): x=1, -1 \leqq y \leqq 1, z=1\}$ which does not belong to the space.

More specifically, if n is odd,

$$S_n = \left\{(x, y, z): \frac{n-1}{n+1} \leqq x \leqq \frac{n+1}{n+3}, \ y = \sin \frac{\pi}{1-x}, \text{ and } z = \frac{n-1}{n+1}\right\}.$$

If n is even,

$$S_n = \left\{(x, y, z): x = \frac{n}{n+2}, \ \frac{n-2}{n} \leqq z \leqq \frac{n}{n+2}, \text{ and } y = \sin \frac{\pi}{1-z}\right\}.$$

This determines an infinite chain of arcs, and it is to be shown that $\{S_n\}$ is locally finite.

If $1>d>0$, there is an N so that $S_n \subset \{(x, y, z): x>1-d \text{ and } z>1-d\}$ if $n>N$. Thus if (x, y, z) is a point of C where $\{S_n\}$ is not locally finite, $x=1$ and $z=1$. In this case, $y=2$, yet S_n is locally finite at $(1, 2, 1)$ because no S_n has a y-coordinate greater than 1. Therefore $\{S_n\}$ is locally finite, and C does not have the f.p.p. Thus $X \times X$ does not have the f.p.p.

EXAMPLE 3. This is a simple example of a separable, locally contractible metric space X that has the f.p.p., yet is not compact. The space X will be a subspace of E_2. Let $X = \bigcup_{n \geqq 0} I_n$ where $I_0 = [(0, 0), (1, 0)]$, a unit interval on the x-axis, and $I_n = [(1/n, 0), (1/n, 1)]$, a vertical unit interval extending up from I_0.

Suppose f is a function from X to X with no fixed point. If f_0 is the restriction of f to the domain I_0, and f_0 is projected onto I_0, a continuous function is obtained from I_0 to I_0. It must have a fixed point $(p, 0)$. Since f_0 has no fixed point, there is an integer k so that $p=1/k$. Thus $f(1/k, 0)=(1/k, y)$ and $0<y \leqq 1$. Let t be the least upper bound of $\{y: f(1/k, y)=(1/k, z) \text{ with } z>y\}$. It can be shown that $(1/k, t)$ is a fixed point. Note that by making I_n, $n>0$, of length n instead of unit length, an unbounded subset of the plane with the f.p.p. may be constructed.

THEOREM 3. *If X is a locally connected, locally compact metric space with the f.p.p., then X is compact* (*see* [**4**, p. 35]).

PROOF. Suppose X is not compact. By a previous lemma, the one point compactification $(X+p)$ is a Peano continuum. It therefore contains an arc $[a, p]$. Using Corollary 14 of [**2**] as in Theorem 2, it follows that X cannot have the f.p.p.

EXAMPLE 4.[1] This is an example of a compact metric space X that does not have the f.p.p. yet contains a dense subset Y that does have the f.p.p.

X is a subspace of E_2. Let A be the square (not including its interior) with $(0, -2)$, $(4, -2)$, $(4, 2)$, and $(0, 2)$ as its four corners. Let $B = \{(x, y): 0 < x \leqq 1, y = \sin 1/x\}$ and $X = A \cup B$. X does not have the f.p.p. Project B into the vertical line joining $(0, -2)$ and $(0, 2)$, and then rotate A 90 degrees. This determines a continuous function from X to X with no fixed point.

Let A' be A minus the open vertical interval from $(0, -1)$ to $(0, 1)$ and Y be $A' \cup B$. It will be shown that Y has the f.p.p. Suppose f is a function from Y to Y with no fixed point. Then $f(A')$ intersects B, because if it did not, there would be a fixed point in A'. Since A' is compact and connected, $f(A')$ is compact and connected and thus contained in B. This implies that there exists a point p of B so that $f(p)$ is in B. If S is a compact, connected set containing p, $f(S) \subset B$. Thus $f(B) \subset B$, and $f[(B) \cup (0, 1)] \subset B$. However, $B \cup (0, 1)$ is homeomorphic to the space in Example 2, and thus f must have a fixed point. It is easy to see that Y is dense in X.

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[1] This example is due to W. L. Strother.