ON TYPICALLY-REAL FUNCTIONS IN A CUT PLANE

BY E. P. MERKES

1. Introduction. Let

\[ w = f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \]

be regular in \(|z| < 1\) and real valued if and only if \(z\) is real and

\(-1 < z < 1\). Then \(f(z)\) is said to be typically-real in the unit circle.
Rogosinski [3] has shown that a necessary and sufficient condition for a regular function \(f(z)\) in \(|z| < 1\), where \(f(0) = 0, f'(0) = 1\), to be typically-real is that

\[ f(z) = \frac{z}{1 - z^2} \phi(z), \quad \phi(0) = 1, \]

where \(\phi(z)\) is regular, real for real \(z\), and has positive real part for

\(|z| < 1\). Furthermore, if \(w = f(z)\) maps each circle \(|z| = r < 1\) into a contour having the property that every line parallel to the imaginary axis cuts this contour in at most two points, then \(f(z)\) is said to be convex in the direction of the imaginary axis. A necessary and sufficient condition that \(f(z)\) be convex in the direction of the imaginary axis is that \(zf'(z)\) be typically-real in the unit circle [1; 2].

It is the main purpose of this paper to display a connection between typically-real functions on one hand and Stieltjes transforms and continued fractions on the other. To this end consider first a function \(F(\xi)\) which is real valued for real \(\xi\) and regular in the complex plane cut along the negative real axis from \(-\infty\) to \(-1\). If, further, \(F(0) = 0, F'(0) = 1, \) and \(\text{Im } F(\xi) \neq 0\) for nonreal values of \(\xi\) in this cut plane, the function \(F(\xi)\) is said to be typically-real in the cut plane. The class of all such functions \(F(\xi)\) is denoted by \(T[-\infty, -1]\). On the other hand, if \(F(0) = 1\) and \(\text{Re } F(\xi) > 0\) for \(\xi\) in this cut plane, we say that \(F(\xi)\) is in the class \(R[-\infty, -1]\).

If the usual agreements are made regarding the normalization, a similar definition can be given for a function to be typically-real in the complex plane cut along the real axis from \(a\) to \(b\) such that the cut does not include both \(0\) and \(\infty\). In each case there is a linear fractional transformation with real coefficients which carries such a function into one which, except for the normalization, is in the class \(T[-\infty, -1]\). For this reason attention is confined to the latter class.

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2. Some characterizations. The univalent transformation

\begin{equation}
\xi = \frac{4z}{(1 - z)^2}
\end{equation}

maps the disc \(|z| < 1\) onto the \(\xi\)-plane cut along the negative real axis from \(\infty\) to \(-1\). Since \(\text{Im } \xi = 4(1 - |z|^2)^{-1} \text{Im } z\), the function

\begin{equation}
f(z) = \frac{1}{4} F \left[ \frac{4z}{(1 - z)^2} \right]
\end{equation}

is typically-real in the unit circle whenever \(F(\xi)\) is in the class \(T[-\infty, -1]\) and conversely. Thus there is a one-to-one correspondence between the class \(T[-\infty, -1]\) and the class of typically-real functions in the unit circle.

By (1.2) and (2.2) we obtain, after the change of variable (2.1), the following restatement of the cited theorem of Rogosinski:

Theorem 2.1. \(F(\xi)\) is in \(T[-\infty, -1]\) if and only if there exists a function \(\Phi(\xi)\) in the class \(R[-\infty, -1]\) such that

\begin{equation}
F(\xi) = \frac{\xi}{(1 + \xi)^{1/2}} \Phi(\xi),
\end{equation}

where that branch of \((1+\xi)^{1/2}\) is chosen in the cut plane which is positive at \(\xi = 0\).

It is known [6, p. 278] that \(\Phi(\xi)\) is in \(R[-\infty, -1]\) if and only if there exists a nondecreasing function \(\alpha(t)\) on \(0 \leq t \leq 1\) such that \(\alpha(1) - \alpha(0) = 1\) and

\begin{equation}
\Phi(\xi) = (1 + \xi)^{1/2} \int_0^1 \frac{d\alpha(t)}{1 + \xi t}.
\end{equation}

Thus we have the following result:

Corollary 2.1. A necessary and sufficient condition for \(F(\xi)\) to be in \(T[-\infty, -1]\) is that there exists a nondecreasing function \(\alpha(t)\) on \(0 \leq t \leq 1\) such that \(\alpha(1) - \alpha(0) = 1\) and

\begin{equation}
F(\xi) = \int_0^1 \frac{\xi d\alpha(t)}{1 + \xi t}.
\end{equation}

By well-known [6, p. 263] relations between the Hausdorff moment problem and \(S\)-fractions, the following corollaries are obtained from the above:
Corollary 2.2. $F(\xi)$ is in $T[-\infty, -1]$ if and only if

$$F(\xi) = \frac{\xi}{1 + \frac{(1 - g_0)g_1\xi}{1 + \frac{(1 - g_n)g_{n+1}\xi}{1 + \cdots}}}$$

where $0 \leq g_n \leq 1$, $n = 0, 1, 2, \ldots$.

Corollary 2.3. Let

$$F(\xi) = \xi - \alpha_2\xi^2 + \alpha_3\xi^3 - \cdots + (-1)^{n+1}\alpha_n\xi^n + \cdots,$$

where $|\xi| < 1$.

Then $F(\xi)$ is in $T[-\infty, -1]$ if and only if the sequence $1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots$ is totally monotone.

By (2.1), (2.2), and (2.6) we obtain, after an equivalence transformation, the following continued fraction characterization of this class:

Theorem 2.2. A necessary and sufficient condition for $f(z)$ to be typically-real in the unit circle is that $f(z)$ have a continued fraction expansion of the form

$$f(z) = \frac{z}{1 - z\left[\frac{1}{1-z} + \frac{4(1-g_0)g_1z}{1-z} + \cdots\right]}$$

(2.8)

$$+ \frac{4(1-g_n)g_{n+1}z}{1-z} + \cdots,$$

where $0 \leq g_n \leq 1$, $n = 0, 1, 2, \ldots$.

The correspondence between the classes under consideration can also be used to characterize typically-real functions in $|z| < 1$ in terms of Schur summability. For this purpose let $g(z) = \sum_{j=0}^\infty c_j(-z)^j$, $G(\xi) = \sum_{j=0}^\infty \gamma_j(-\xi)^j$ be related by

$$\frac{1}{2} (1-z)g(z) = G(\xi), \quad \xi = 4z/(1-z)^2.$$  

(2.9)

The transformation from the sequence $s_n = \sum_{j=0}^n c_j$ to the sequence $S_n = \sum_{j=0}^n \gamma_j$ by means of the identity (2.9) is called the ($S$)-transformation. The sequence $\{s_n\}$ is said to be ($S$)-summable to the value $S$ if $S = \lim_{n \to \infty} S_n$. Scott and Wall [4] have introduced and studied in detail this consistent method of summability.

Theorem 2.3. Let $f(z) = \sum_{j=0}^\infty a_jz^j$, $a_0 = 0$, $a_1 = 1$, be regular and real for real $z$ in the unit circle. Then $f(z)$ is typically-real in $|z| < 1$ if
and only if the sequence \( \{1 + (-1)^n(a_{n+2} - a_n)/2\}_{n=0}^\infty \) is \((S)\)-summable to a value not exceeding unity and such that \( \{S_n - S_{n-1}\}_{n=0}^\infty \), where \( S_{-1} = 0 \) and the sequence \( S_n \) is the \((S)\)-transform of the given sequence, is totally monotone.

**Proof.** By (2.4) and the characterization of Rogosinski \( f(z) \) is typically-real in the unit circle if and only if

\[
\frac{(1 - z)^2}{4z} \left[ \frac{(1 + z)^2}{z} f(z) - 1 \right] = \frac{(1 - z)^2}{4z} \left[ \frac{1 + z}{1 - z} \phi(z) - 1 \right] = \frac{1}{\zeta} \left[ (1 + \zeta)^{1/2} \Phi(\zeta) - 1 \right] = \frac{1}{\zeta} \left[ (1 + \zeta) \int_0^1 \frac{d\alpha(t)}{1 + \zeta t} - 1 \right] = \int_0^1 \frac{(1 - t)d\alpha(t)}{1 + \zeta t},
\]

where \( \zeta = 4z/(1-z)^2 \), \( \Phi(\zeta) = \phi(z) \), and \( \alpha(1) - \alpha(0) = 1 \). Let \( G(\zeta) \) be the function represented by the integral in (2.10). Then \( \{\gamma_j\} \) is totally monotone and \( \sum_{j=0}^\infty \gamma_j \leq 1 \) where \( G(\zeta) = \sum_{j=0}^\infty \gamma_j (\zeta/\gamma_j)^j \) for \( |\zeta| < 1 \) [6, p. 284]. Moreover, if the left-hand side of (2.10) is taken to be the function \((1 - z)g(z)/2 \) in (2.9), it is easily seen that \( s_n = \sum_{j=0}^\infty c_j = 1 + (-1)^n(a_{n+2} - a_n)/2 \). The theorem now follows at once from the definition of \((S)\)-summability.

3. **Some properties of the class \( T[-\infty, -1] \).** Among other considerations the following theorem yields some properties of \( S \)-fractions of the form (2.6):

**Theorem 3.1.** If \( F(\zeta) \) is in \( T[-\infty, -1] \), then it is univalent for \( \text{Re } \zeta > -1 \). This result is sharp. Moreover, \( F(\zeta) \) is convex in the direction of the imaginary axis for \( |\zeta| < 1 \).

The domain of univalence, \( \text{Re } \zeta > -1 \), for the class is obtained by a trivial adjustment of a proof given by Thale [5, p. 233]. That the result is sharp is established by considering the following functions of \( T[-\infty, -1] \):

\[
F(\zeta; c) = \frac{\zeta(1 + c\zeta)}{1 + \zeta}, \quad 0 < c < 1.
\]

It is easily seen that for any two distinct points \( \zeta_1 \) and \( \zeta_2 \) in the upper (lower) half-plane and on the line \( \text{Re } \zeta = -1 \) there is a constant \( c_0, 0 < c_0 < 1 \), such that \( F(\zeta_1; c_0) = F(\zeta_2; c_0) \).

In order to prove that \( F(\zeta) \) is convex in the direction of the imaginary axis, note that by (2.5)
(3.2) \[ \text{Im} [\zeta F'(\zeta)] = \text{Im} \zeta \int_0^1 \frac{1 - |\zeta|^2 t^2}{1 + |\zeta t|^4} \, d\alpha(t). \]

It immediately follows that \( \zeta F'(\zeta) \) is typically-real. This yields the desired result.

It is interesting to note [5] that the domain of univalence for the class \( T[-\infty, -1] \) in Theorem 3.1 is also a sharp domain of univalence for functions of the form \( F(\zeta)/\zeta \), where \( F(\zeta)(\neq \zeta) \) is in \( T[-\infty, -1] \).

One way to see this is to observe that by (2.7)

(3.3) \[ F(\zeta)/\zeta = 1 - \alpha_2 F_1(\zeta). \]

By a property of totally monotone sequences it now follows from Corollary 2.3 that \( F_1(\zeta) \) is in \( T[-\infty, -1] \). The univalence of \( F_1(\zeta) \) then implies that of \( F(\zeta)/\zeta \).

4. A domain of univalence for typically-real functions in the unit circle. Let \( f(z) \) be typically-real in the unit circle. By the one-to-one correspondence of \$2 \$ there exists a function \( F(\zeta) \) in \( T[-\infty, -1] \) such that (2.2) holds. From Theorem 3.1 and the fact that (2.1) is a univalent transformation, we conclude that \( f(z) \) is univalent for

(4.1) \[ \text{Re} \left[ \frac{4z}{(1 - z)^2} \right] > -1. \]

The region (4.1) can be expressed in the form given in the following result:

**Theorem 4.1.** If \( f(z) \) is typically-real in the unit circle, then \( f(z) \) is univalent for \( z \) in the domain \( D \) bounded by the circular arcs \( z=r e^{i\theta} \), where

(4.2) \[ r = (1 + \sin^2 \theta)^{1/2} - |\sin \theta|, \quad 0 \leq \theta < 2\pi. \]

The result is sharp in the sense that any open region of univalence for this class of functions which contains \( D \) is coincident with \( D \).

The sharpness result of the theorem follows from that for the class \( T[-\infty, -1] \).

The largest circular domain with center at the origin and contained in the domain \( D \) has radius \((2)^{1/2} - 1\).

**Corollary 4.1.** If \( f(z) \) is typically-real in the unit circle, then \( f(z) \) is univalent in the disc \( |z| < (2)^{1/2} - 1 \). This circular domain of univalence cannot be replaced by a larger circular domain with center at the origin in which each function of the class under consideration is univalent.

The function
(4.3) \[ f(z) = z(1 + z^2)/(1 - z^2)^2 \]
is typically-real in the unit circle. Moreover, since the derivative of
this function vanishes at \( z = \pm ((2)^{1/2} - 1)i \), (4.3) is not univalent in
any larger circular domain with center at the origin than that given
in Corollary 4.1.

References

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ON THE POLE AND ZERO LOCATIONS OF RATIONAL
LAPLACE TRANSFORMATIONS OF NON-NEGATIVE
FUNCTIONS

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Let \( a(t) \) be a real, bounded function of the real variable \( t \) defined
in the interval, \( 0 \leq t < \infty \), and let its Laplace-Stieltjes transform,

(1) \[ F(s) = \int_0^\infty e^{-st} da(t), \]

be a rational function of the complex variable \( s = \sigma + i\omega \), having at
least as many poles as zeros. \( F(s) \) may be written as

(2) \[ F(s) = \frac{\prod_{i=1}^h (s - \eta_i) \prod_{i=1}^q (s - \nu_i)}{\prod_{i=1}^m (s - \rho_i) \prod_{i=1}^q (s - \xi_i)} \]

where the \( \eta_i \) and \( \rho_i \) are real and the \( \nu_i \) and \( \xi_i \) are complex. Under these

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