A FUNCTIONAL EQUATION PROPOSED BY R. BELLMAN

E. G. STRAUS

R. Bellman (Bull. Amer. Math. Soc. vol. 64 (1958) p. 178, problem 14) has posed (essentially) the following problem:

Let $f(u_0, u_1, \cdots, u_n)$ be analytic for $u_0 \neq 0$ and satisfy the functional equation

$$(1) \quad f(uv, (uv)', \cdots, (uv)^{(n)}) = f(u, u', \cdots, u^{(n)}) + f(v, v, \cdots, v^{(n)})$$

for arbitrary x and arbitrary nonzero n times differentiable functions $u(x)$, $v(x)$. What is the form of f for general n?

At the suggestion of R. Bellman and of N. J. Fine (who has given another solution of this problem),[1] we give here a solution which seems to stress the proper setting of the problem.

Set $u = e^s$, $v = e^t$; then we can write

$$f(u, u', \cdots, u^{(n)}) = g(s, s', \cdots, s^{(n)}),$$

and the functional equation (1) becomes

$$(2) \quad g(s + t, s' + t', \cdots, s^{(n)} + t^{(n)}) = g(s, \cdots, s^{(n)}) + g(t, \cdots, t^{(n)})$$

for arbitrary x, and arbitrary n times differentiable s, t.

Applying (2) to the functions

$$s = \sum_{k=0}^{n} \frac{s_k(x - x_0)^k}{k!}$$

we get for $G(s) = g(s, s', \cdots, s^{(n)})$ that at $x = x_0$

$$(3) \quad \begin{aligned} G(s) &= G\left(\sum \frac{s_k(x - x_0)^k}{k!}\right) = \sum G\left(\frac{s_k(x - x_0)^k}{k!}\right) \\ &= g(s_0, 0, \cdots, 0) + g(0, s_1, 0, \cdots, 0) + \cdots \\ &\quad + g(0, 0, \cdots, 0, s_n) \\ &= \sum_{k=0}^{n} g_k(s_k). \end{aligned}$$

As a result of (2) we see that each of the functions $g_k(s_k)$ is additive. Summing up we have the following

Received by the editors December 29, 1958 and, in revised form, February 2, 1959.

[1] Solutions have also been given by W. F. Trench and by J. Aczél and M. Hosszú.

THEOREM 1. *Let $f(x, u_0, u_1, u_2, \cdots, u_n)$ be a function of $n+2$ variables so that*

$$f(x, uv, (uv)', \cdots, (uv)^{(n)}) = f(x, u, u', \cdots, u^{(n)}) + f(x, v, v', \cdots, v^{(n)}) \tag{4}$$

for all x and all nonzero n times differentiable u, v. Then

$$f(x, u, u', \cdots, u^{(n)}) = \sum_{k=0}^{n} f_k(x, d^k \log u/dx^k)$$

where each $f_k(x, y)$ is additive in y. In particular, under any one of the usual conditions (e.g., continuity, measurability, boundedness) on $f_k(x, y)$ as a function of y we obtain

$$f(x, u', \cdots, u^{(n)}) = \sum_{k=0}^{n} a_k(x) \frac{d^k \log u}{dx^k}. \tag{5}$$

In Bellman's problem f did not directly depend on x so that the a_k in (5) would be constant.

We can now make several observations. The first (whose formulation is due to R. Steinberg) is that the function uv in (1) and (4) could be replaced by $\psi(u, v) = \phi_1(\phi_2(u) + \phi_2(v))$, where ϕ_i is an n times differentiable function and $\phi_2' \neq 0$, where the range of ϕ_2 is the entire real line. So we have the following

THEOREM 2. *Let $f(x, u_0, u_1, \cdots, u_n)$ be a function of $n+2$ variables so that*

$$f(x, \psi(u, v), (\psi(u, v))', \cdots, (\psi(u, v))^{(n)}) = f(x, u, \cdots, u^{(n)}) + f(x, v, \cdots, v^{(n)}). \tag{6}$$

Then

$$f(x, u, u', \cdots, u^{(n)}) = \sum_{k=0}^{n} f_k\left(x, \frac{d^k \phi_2(u)}{dx^k}\right) + a(x) \tag{7}$$

where $f_k(x, y)$ is additive in y, and under suitable regularity conditions

$$f(x, u, u', \cdots, u^{(n)}) = \sum_{k=0}^{n} a_k(x) \frac{d^k \phi_2(u)}{dx^k} + a(x). \tag{7'}$$

PROOF. We set $s = \phi_2(u)$, $t = \phi_2(v)$. Then we can write

$$f(x, u, u', \cdots, u^{(n)}) = g(x, s, \cdots, s^{(n)}) (= g(x, s) \text{ for short})$$

and

$$f(x, \phi_1(s), \cdots, \phi_1(s)^{(n)}) = h(x, s, \cdots, s^{(n)}) (= h(x, s) \text{ for short}).$$

Thus (6) becomes

$$(6') \quad h(x, s+t, \cdots, s^{(n)} + t^{(n)}) = g(x, s, \cdots, s^{(n)}) + g(x, t, \cdots, t^{(n)})$$

for all n times differentiable functions s, t at a point x.

For $t=0$ we obtain

$$h(x, s) = g(x, s) + g(x, 0) = g(x, s) + a(x)$$

and for

$$k(x, s) = g(x, s) - a(x) = h(x, s) - 2a(x)$$

equation (6′) becomes

$$(6'') \qquad k(x, s+t) = k(x, s) + k(x, t).$$

This equation is analogous to (2) and leads to (7). We can now use (7) to analyze the functions ϕ_1 and a to obtain

$$\phi_1(u) = \phi_2^{-1}(u + c) \quad \text{and} \quad a(x) = f_0(x, c)$$

for some constant c.

Finally the fact that differentiation operators only were used was merely due to the fact they satisfy relations of the form

$$D^k f(s) = g(s, Ds, D^2 s, \cdots, D^k s)$$

for all k times differentiable s and arbitrarily given k times differentiable f. We could replace these D^k by arbitrary linear operators $T_1, \cdots, T_n$ which satisfy identities of the form

$$(8) \qquad (T_i \phi_1(s))(x) = g_i(s, T_1 s, \cdots, T_n s) \qquad i = 1, \cdots, n$$

for all x, all s in the prescribed function space and the function ϕ_1 as in Theorem 2.

It is quite unnecessary to restrict the operator $f(u, u', \cdots, u^{(n)})$ to dependence on a finite number of derivatives; the same result would hold if we had a differential operator of infinite order, or more generally, if we had operators T_i satisfying (8) and so that for every x there exist functions s_i with $(T_i s_j)(x) = \delta_{ij}$.

We forego the statement of this theorem in all its gruesome generality.

UNIVERSITY OF CALIFORNIA, LOS ANGELES