I. Trigonometric polynomials with coefficients ±1. Consider the trigonometric polynomial

\[ P(e^{i\theta}) = \sum_{n=1}^{N} \epsilon_n e^{in\theta} \]

where \( \epsilon_n = \pm 1 \). If we set \( \|P\|_\infty = \max_{\theta} |P(e^{i\theta})| \), the Parseval theorem shows that \( \|P\|_\infty \geq N^{1/2} \), and the following problem arises: does there exist an absolute constant \( A \) with the property that for each \( N \) one can find \( \epsilon_1, \ldots, \epsilon_N \), equal to \( \pm 1 \), so that

\[ \|P\|_\infty \leq AN^{1/2}, \]

where \( P \) is given by (1.1)?

If one allows the coefficients \( \epsilon_n \) to be complex numbers of absolute value 1, an affirmative answer to the question is furnished by the partial sums of the series \( \sum_1^\infty e^{in\log e^{in\theta}} \); this example is due to Hardy and Littlewood [4, pp. 116–118]. A theorem of Salem and Zygmund [2, pp. 270, 278] shows, roughly speaking, that \((N \log N)^{1/2}\) is the “most probable” order of magnitude for \( \|P\|_\infty \) if \( \epsilon_n = \pm 1 \).

During the summer of 1958, Salem drew my attention to the problem stated in the first paragraph. It turns out that an affirmative answer can be given by a construction which uses nothing more sophisticated than the parallelogram law

\[ |\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2. \]

After I found this construction I learned that the problem had been solved earlier, by essentially the same method, in the 1951 Master’s Thesis of H. S. Shapiro [3]. Since the result is needed in Part II of this paper, I am publishing the proof here, with Shapiro’s consent. As in the Hardy-Littlewood example, the polynomials (1.1) may actually be taken as the partial sums of a fixed series \( \sum_1^\infty \epsilon_n e^{in\theta} \):

**Theorem I.** There exists a sequence \( \{\epsilon_n\} (n = 1, 2, 3, \ldots) \), with \( \epsilon_n = 1 \) or \(-1\), such that

\[ \left| \sum_{n=1}^{N} \epsilon_n e^{i n \theta} \right| < 5N^{1/2} \quad (0 \leq \theta < 2\pi; \quad N = 1, 2, 3, \ldots). \]

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Proof. Set $P_0(x) = Q_0(x) = x$, and define polynomials $P_k$ and $Q_k$ inductively by

$$
\begin{align*}
(P_{k+1}(x) &= P_k(x) + x^{2k} Q_k(x), \\
Q_{k+1}(x) &= P_k(x) - x^{2k} Q_k(x),
\end{align*}
\tag{1.5}
$$

Then $P_k(e^{i\theta})$ is of the form (1.1), with $N = 2^k$, and $P_k$ is a partial sum of $P_{k+1}$. Hence we can define a sequence $\{\epsilon_n\}$ by setting $\epsilon_n$ equal to the $n$th coefficient of $P_k$, where $2^k > n$; this sequence will be shown to have the desired properties.

For $|x| = 1$, (1.3) and (1.5) imply

$$
|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 = |P_k(x) + x^{2k} Q_k(x)|^2 + |P_k(x) - x^{2k} Q_k(x)|^2
= 2|P_k(x)|^2 + 2|Q_k(x)|^2,
$$

and since $|P_0(x)|^2 + |Q_0(x)|^2 = 2$, we conclude that

$$
|P_k(e^{i\theta})|^2 + |Q_k(e^{i\theta})|^2 = 2^{k+1}.
\tag{1.6}
$$

Hence

$$
|P_k(e^{i\theta})| \leq 2^{1/2} \cdot 2^{k/2},
\tag{1.7}
$$

which proves (1.4) for $N = 2^k$.

If now $s_n(P_k)$ and $s_n(Q_k)$ denote the $n$th partial sums of $P_k$ and $Q_k$ respectively, where $1 \leq n \leq 2^k$, then

$$
\begin{align*}
\left| s_n(P_k)(e^{i\theta}) \right| &\leq (2 + 2^{1/2}) 2^{k/2} \\
\left| s_n(Q_k)(e^{i\theta}) \right| &\leq (2 + 2^{1/2}) 2^{k/2}
\end{align*}
\tag{1.8}
$$

This is obviously true if $k = 0$. Suppose (1.8) holds for some $k$, and consider $s_n(P_{k+1})$ and $s_n(Q_{k+1})$, with $1 \leq n \leq 2^{k+1}$. If $n \leq 2^k$, (1.5) shows that

$$
\left| s_n(P_{k+1}) \right| = \left| s_n(Q_{k+1}) \right| = \left| s_n(P_k) \right| < (2 + 2^{1/2}) 2^{(k+1)/2}.
$$

If $2^k < n \leq 2^{k+1}$, (1.5) and (1.7) show that

$$
\begin{align*}
\left| s_n(P_{k+1}) \right| &\leq \left| P_k \right| + \left| s_{n-2}(Q_k) \right| \\
&\leq 2^{(k+1)/2} + (2 + 2^{1/2}) 2^{k/2} = (2 + 2^{1/2}) 2^{(k+1)/2}.
\end{align*}
$$

The same estimate holds for $\left| s_n(Q_{k+1}) \right|$, and (1.8) is proved by induction.

To complete the proof of (1.4), suppose $2^k - 1 \leq N \leq 2^k$. By (1.8), we have

$$
\left| s_N(P_k)(e^{i\theta}) \right| \leq (2 + 2^{1/2}) 2^{k/2} \leq 2(1 + 2^{1/2}) N^{1/2} < 5N^{1/2}.
$$

II. Transformations of Fourier coefficients. In this section, $p$ and
q will always denote conjugate exponents, i.e., \(1/p + 1/q = 1\). For \(1 \leq p < \infty\), \(L^p\) denotes the usual Lebesgue space of complex functions on the unit circle, normed by

\[
\|f\|_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right\}^{1/p}.
\]

\(L^\infty\) is the space of all essentially bounded measurable functions on the circle. The Fourier coefficients of any \(f \in L^1\) will be denoted by

\[
f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

If \(F\) is a complex function defined in the plane, we say that \(F\) maps \(A\) into \(B\), where \(A\) and \(B\) are function spaces on the circle, if to every \(f \in A\) there corresponds a \(g \in B\) (we shall write: \(g = F \circ f\)) such that \(g = F(f)\). In other words, it is required that the series \(\sum c_n e^{int}\) should be the Fourier series of a function in \(B\) whenever \(\sum c_n e^{int}\) is the Fourier series of a function in \(A\).

The functions \(F\) which map \(L^1\) into \(L^1\) have recently been determined \([1]\); they are precisely those which are real-analytic near the origin (i.e., in some neighborhood of the origin); of course we must also have \(F(0) = 0\). For the other Lebesgue spaces, the situation is quite different. We first state some sufficient conditions:

**Theorem II.** Suppose \(1 < p \leq 2\), and suppose there is a constant \(A\) such that \(|F(z)| \leq A |z|^{q/2}\) near the origin. Then \(F\) maps \(L^p\) into \(L^2\).

**Proof.** If \(f \in L^p\), the Hausdorff-Young theorem \([4, p. 190]\) shows that \(\sum |\hat{f}(n)|^q < \infty\), so that \(\sum |F(\hat{f}(n))|^2 < \infty\).

**Theorem III.** Suppose \(1 \leq p \leq 2\). If \( |F(z)| \leq A |z|^{2/p}\) near the origin, then \(F\) maps \(L^q\) into \(L^2\).

**Proof.** If \(f \in L^q\), then \(\sum |\hat{f}(n)|^2 < \infty\), so that \(\sum |F(\hat{f}(n))|^p < \infty\), and the Hausdorff-Young theorem implies that \(F \circ f \in L^q\).

**Remarks.** 1. For \(q = 2\), this condition is necessary as well as sufficient.

2. For \(q = \infty\), the hypothesis of Theorem III is: \( |F(z)| \leq A |z|^2\). It follows that \(F\) maps \(L^\infty\) (even \(L^2\)) into the class of functions which are sums of absolutely convergent trigonometric series.

3. If \(F\) is of the form

\[
F(z) = a_1 z + a_2 \bar{z} + |z|^{2/p} b(z),
\]

where \(b\) is a function which is bounded near the origin, then \(F\) also maps \(L^q\) into \(L^q\). Note that no smoothness conditions are imposed on
b (not even measurability is needed), in strong contrast to the results in [1].

I do not know whether (2.3) holds whenever $F$ maps $L^q$ into $L^q$. However, if we restrict ourselves to even functions $F$, Theorem I can be used to show that Theorem III states a condition which is necessary as well as sufficient. In fact, the following stronger assertion holds:

**Theorem IV.** Suppose $1 \leq p < \infty$, $F$ is an even function, and $|z|^{-2/p} |F(z)|$ is not bounded near the origin. Then there is a continuous function $f$ on the circle to which corresponds no $g \in L^q$ with $\hat{g} = F(\hat{f})$.

In other words, $F$ does not map the space of all continuous functions into $L^q$, hence it does not map $L^q$ into $L^q$.

**Proof.** The hypothesis implies the existence of numbers $z_m \neq 0$ ($m = 1, 2, 3, \cdots$), such that $m^2 z_m \to 0$ and $|F(z_m)| > m^5 |z_m|^{2/p}$. Define $N_m = [m^{-4} z_m^{-2}]$. These choices produce the relations

\[
\sum_{m=1}^{\infty} |z_m| N_m^{1/2} < \infty
\]

and

\[
|F(z_m)| N_m^{1/p} \to \infty \quad \text{as} \quad m \to \infty.
\]

Now choose integers $n_m$ so that

\[
n_m + N_m < n_{m+1} - N_{m+1}
\]

and define

\[
T_m(e^{i\theta}) = z_m e^{i n_m \theta} (e_1 e^{i \theta} + \cdots + e_{N_m} e^{i N_m \theta}),
\]

where $\{e_n\}$ is the sequence of Theorem I. The series

\[
f(e^{i\theta}) = \sum_{m=1}^{\infty} T_m(e^{i\theta})
\]

converges uniformly, by (2.4) and Theorem I, so that $f$ is continuous.

Define the kernels $K_m$ by

\[
K_m(e^{i\theta}) = e^{i(n_m+N_m)\theta} \sum_{n=-2N_m}^{2N_m} \min \left(1, 2 - \frac{|n|}{N_m} \right) e^{in\theta}.
\]

Suppose there is a function $g \in L^q$ such that $\hat{g} = F(\hat{f})$, i.e., $g = F \circ f$. Our choice of $\{n_m\}$ implies that $g \ast K_m = F \circ T_m$, where

\[
(g \ast K_m)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i(\theta-\phi)}) K_m(e^{i\phi}) d\phi.
\]
Since $\|K_m\|_1 < 3$, we see that
\begin{equation}
\|F \circ T_m\|_q < 3\|g\|_q \quad (m = 1, 2, 3, \ldots).
\end{equation}

On the other hand, the assumption that $F(-z_m) = F(z_m)$ shows that
\begin{equation}
(F \circ T_m)(e^{i\theta}) = F(z_m)e^{in_m \theta}(e^{i\theta} + \cdots + e^{iN_m \theta}),
\end{equation}
so that
\begin{equation}
\left| (F \circ T_m)(e^{i\theta}) \right| = \left| F(z_m) \right| \cdot \frac{\sin (N_m \theta/2)}{\sin (\theta/2)}.
\end{equation}

An easy computation now yields
\begin{equation}
\|F \circ T_m\|_q > C_q \left| F(z_m) \right| N_m^{1/p},
\end{equation}
where $C_q$ is a positive constant, depending only on $q$. By (2.5), (2.14) implies that $\|F \circ T_m\|_q \rightarrow \infty$ as $m \rightarrow \infty$, and this contradicts (2.11).

The theorem follows.

References


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