

## HAUSDORFF TRANSFORMS OF BOUNDED SEQUENCES

J. H. WELLS

Let  $\phi$  be a complex valued function of bounded variation on  $[0, 1]$  such that  $\phi(0) = 0$ . The statement that  $H(\phi)$  is the *Hausdorff transformation* generated by  $\phi$  means that if  $x$  is a complex number sequence, then  $y = H(\phi)x$  where

$$y_n = \int_0^1 \sum_{p=0}^n C_{n,p} t^p (1-t)^{n-p} x_p d\phi(t) \quad (n = 0, 1, 2, \dots).$$

The sequence is said to be  $H(\phi)$  evaluable to  $L$  if  $y_n \rightarrow L$  as  $n \rightarrow \infty$ . The transformation  $H(\phi)$  is *regular* (i.e.,  $\lim x_n = L$  implies  $\lim y_n = L$ ) if and only if  $\phi(0+) = 0$  and  $\phi(1) = 1$ . If  $x$  is a complex number sequence, then  $C(x)$  denotes the set of limit points of  $x$ .

Among the regular Hausdorff transformations are the methods  $C_r$  ( $\text{Re } r > 0$ ) of Cesàro,  $H_r$  ( $\text{Re } r > 0$ ) of Hölder and  $E_r$  ( $0 < r < 1$ ) of Euler for which  $\phi(t) = 1 - (1-t)^r$ ,

$$\phi(t) = \frac{1}{\Gamma(r)} \int_0^t (\log 1/u)^{r-1} du, \text{ and } \phi(t) = \begin{cases} 0, & 0 \leq t < r, \\ 1, & r \leq t < 1 \end{cases}$$

respectively. It has been shown by Barone [2] that the regular Hausdorff transformations  $C_r$  ( $\text{Re } r > 0$ ),  $H_r$  ( $r$  a positive integer) and  $E_r$  ( $0 < r < 1$ ) have the property that the set of limit points of the transform of each bounded complex number sequence is connected. In each case the result was obtained by use of

**THEOREM A.** *If  $y$  is a bounded complex number sequence such that  $|y_n - y_{n-1}| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $C(y)$  is connected.*

The object of this note is to prove the following

**THEOREM.** *If  $H(\phi)$  is a regular Hausdorff transformation, then the following two statements are equivalent:*

- (i)  $\phi(1^-) = \phi(1)$ ,
- (ii)  $C(H(\phi)x)$  is connected for each bounded complex number sequence  $x$ .

**1. Proof that (i) implies (ii).** If  $x$  is a bounded complex number sequence, then  $y = H(\phi)x$  is bounded and for each positive integer  $n$

---

Presented to the Society, April 25, 1959; received by the editors March 30, 1959.

$$|y_n - y_{n-1}| \leq M \int_0^1 \sum_{p=0}^n |C_{n,p}t^p(1-t)^{n-p} - C_{n-1,p}t^p(1-t)^{n-1-p}| |d\phi(t)|$$

where  $M = \text{l.u.b. } |x_p|$  and  $C_{n-1,n}t^n(1-t)^{-1} = 0$  ( $0 \leq t \leq 1$ ). If  $b_n(t)$  denotes the largest integer which does not exceed  $nt$ , we have [2, p. 751]

$$|y_n - y_{n-1}| \leq 2M \int_0^1 C_{n-1,b_n(t)}t^{b_n(t)+1}(1-t)^{n-1-b_n(t)} |d\phi(t)|.$$

Let  $\epsilon > 0$ . Since  $\phi(0^+) = \phi(0)$  and  $\phi(1^-) = \phi(1)$  we can choose  $\delta > 0$  so that

$$2M \int_0^\delta |d\phi(t)| < \epsilon/3 \quad \text{and} \quad 2M \int_{1-\delta}^1 |d\phi(t)| < \epsilon/3.$$

An application of Stirling's formula shows that there exists a positive number  $k$  such that, if  $1 < n\delta$ ,

$$C_{n-1,b_n(t)}t^{b_n(t)+1}(1-t)^{n-1-b_n(t)} < k \frac{(n-1)^{n-1/2}}{(n-1/\delta)^n} \quad (\delta \leq t \leq 1-\delta).$$

Consequently, if  $v$  denotes the total variation of  $\phi$  on  $[0, 1]$ ,

$$|y_n - y_{n-1}| \leq \epsilon/3 + 2Mvk \frac{(n-1)^{n-1/2}}{(n-1/\delta)^n} + \epsilon/3 < \epsilon$$

for  $n$  sufficiently large. Thus  $|y_n - y_{n-1}| \rightarrow 0$  as  $n \rightarrow \infty$  so that, by Theorem A,  $C(y)$  is connected. Hence (i) implies (ii).

**2. Proof that (ii) implies (i).** Suppose there exists a regular Hausdorff transformation  $H(\phi)$  with  $\phi(1^-) \neq \phi(1)$  such that  $C(H(\phi)x)$  is connected for each bounded complex number sequence  $x$ . It is sufficient to assume  $\phi$  real valued. Let  $\phi = \phi_1 + \phi_2$  where

$$\phi_1(t) = \begin{cases} \phi(t), & 0 \leq t < 1, \\ \phi(1^-) & t = 1, \end{cases} \quad \phi_2(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \phi(1) - \phi(1^-), & t = 1 \end{cases}$$

and for each pair  $n, p$  of nonnegative integers,  $p \leq n$ , let

$$A_{np} = \int_0^1 C_{n,p}t^p(1-t)^{n-p}d\phi_1(t).$$

It is not difficult to show [3, pp. 308-309] that (1)  $A_{np} \rightarrow 0$  as  $n \rightarrow \infty$  ( $p = 0, 1, 2, \dots$ ). If  $n$  is a nonnegative integer and  $\epsilon > 0$ , then

$$\begin{aligned} |A_{n+p,p}| &\leq (n+p)^n \int_0^\delta t^p |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| \\ &\leq (n+p)^n \delta^p \int_0^\delta |d\phi_1(t)| + \int_\delta^1 |d\phi_1(t)| < \epsilon \end{aligned}$$

for  $p$  sufficiently large, provided  $\delta$  is a number in  $(0, 1)$  so chosen that

$$\int_\delta^1 |d\phi_1(t)| < \epsilon/2.$$

Hence (2)  $A_{n+p,p} \rightarrow 0$  as  $p \rightarrow \infty$  ( $n=0, 1, 2, \dots$ ). Suppose each of  $n$  and  $p$  is a positive integer. Since the function  $C_{n+p,p} t^p (1-t)^n$  is monotone on each of the intervals  $[0, p/(n+p)]$  and  $[p/(n+p), 1]$  we have, after an integration by parts, the inequality

$$\begin{aligned} |A_{n+p,p}| &\leq \int_0^{p/(n+p)} |\phi_1(t)| d[C_{n+p,p} t^p (1-t)^n] \\ (3) \quad &+ \int_{p/(n+p)}^1 |\phi_1(t)| d[-C_{n+p,p} t^p (1-t)^n] \\ &\leq 2MC_{n+p,p} [p/(n+p)]^p [n/(n+p)]^n < k(1/n + 1/p)^{1/2} \end{aligned}$$

where  $M = \text{l.u.b.}_{0 \leq t \leq 1} |\phi_1(t)|$  and  $k$  is a constant arising from an application of Stirling's formula.

From conditions (1), (2) and (3) it follows that  $A_{np} \rightarrow 0$  as  $n \rightarrow \infty$  uniformly with respect to  $p$ . According to a result of Agnew [1] this property of the numbers  $A_{np}$  is sufficient to ensure the existence of a divergent sequence  $x$  of zeros and ones such that  $H(\phi_1)x$  has limit zero. Therefore  $y = H(\phi)x = H(\phi_1)x + H(\phi_2)x$  is the sum of a sequence with limit zero and a divergent sequence each element of which is zero or  $\phi_2(1)$ . Consequently,  $C(y)$  is not connected. This is a contradiction. Hence (ii) implies (i).

#### REFERENCES

1. R. P. Agnew, *A simple sufficient condition that a method of summability be stronger than convergence*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 128-132.
2. H. G. Barone, *Limit points of sequences and their transforms by methods of summability*, Duke Math. J. vol. 5 (1939) pp. 740-752.
3. H. S. Wall, *Analytic theory of continued fractions*, New York, 1948.

THE UNIVERSITY OF NORTH CAROLINA