SUMS OF STATIONARY RANDOM VARIABLES[1]

ENDERS A. ROBINSON

A sequence $x(t)$ $(-\infty<t<\infty$, t an integer) of elements in Hilbert space is called stationary if the inner product $(x(t+s), x(t))$ does not depend upon t. If the Hilbert space is L^2 space with probability measure, then $x(t)$ is a random variable and the sequence $x(t)$ $(-\infty<t<\infty)$ is called a second-order stationary random process. Let $\boldsymbol{X}$ be the closed linear manifold spanned by all the elements of the stationary process. Then Kolmogorov [**1**] has shown that the equation $x(t)U=x(t+1)$, $-\infty<t<\infty$, uniquely determines the unitary operator U with domain and range $\boldsymbol{X}$. Using the von Neumann [**2**] spectral representation of U, we obtain the spectral representation of the random process

$$x(t) = \int_{-.5}^{.5} e^{2\pi i u t} dx(0)E(u), \qquad -\infty < t < \infty.$$

The von Neumann [**3**] ergodic theorem, in the framework of Khintchine [**4**], is applicable, and shows that the average $\sum_1^n x(t)/n$ converges in the mean to the random variable $x(0)[E(0+)-E(0-)]$ as $n\rightarrow\infty$. In this paper we consider sums instead of averages; that is, we consider $\sum_1^n x(t)$, and establish the following theorem.

THEOREM. *Let the random variables $x(t)$ $(-\infty<t<\infty$, t an integer) be a second-order stationary random process with spectral distribution function $F(u)$. For* variance $\{\sum_1^n x(t)\}$ *to be bounded for all positive integers n, each of the following two conditions is necessary and sufficient*:

$$\int_{-.5}^{.5} \sin^{-2} \pi u dF(u) < \infty. \tag{1}$$

(2) *There is a second-order stationary random process*

$$y(t) \ (-\infty<t<\infty) \textit{ satisfying } y(t)-y(t+1)=x(t).$$

PROOF. (NECESSARY CONDITIONS). We are given that *variance* $\{\sum_1^n x(t)\}<B$ for all positive integers n. Without loss of generality we assume that the $x(t)$ are centered so that their mean values are zero. Then

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$$\text{variance}\left\{\sum_1^n x(t)\right\} = \left(\sum_1^n x(t),\ \sum_1^n x(t)\right).$$

From the spectral representation we have

$$\sum_1^n x(t) = \int_{-.5}^{.5} e^{2\pi iu}\,\frac{1 - e^{2\pi iun}}{1 - e^{2\pi iu}}\,dx(0)E(u),$$

so

$$\text{variance}\left\{\sum_1^n x(t)\right\} = \int_{-.5}^{.5} \frac{\left|1 - e^{2\pi iun}\right|^2}{\left|1 - e^{2\pi iu}\right|^2}\,dF(u)$$

where $F(u) = \|x(0)E(u)\|^2$, $-.5 \leqq u \leqq .5$, is the spectral distribution function. Hence we have

$$B > \text{variance}\left\{\sum_1^n x(t)\right\} = \left\{\int_{-.5}^{0-} + \int_{0+}^{.5}\right\} \frac{\sin^2 \pi un}{\sin^2 \pi u}\,dF(u) + n^2[F(0+) - F(0-)]$$

which shows that $F(0+) - F(0-)$ must vanish. Moreover, we have

$$B > \frac{1}{N}\sum_{n=1}^N \left\{\int_{-.5}^{0-} + \int_{0+}^{.5}\right\} \frac{\sin^2 \pi un}{\sin^2 \pi u}\,dF(u)$$
$$= \left\{\int_{-.5}^{0-} + \int_{0+}^{.5}\right\} \left[\frac{1}{N}\sum_{n=1}^N \left(\frac{1}{2} - \frac{1}{2}\cos 2\pi un\right)\right] \sin^{-2} \pi u dF(u).$$

Clearly the limit of the expression in brackets, as $N \to \infty$, is $1/2$, so $\int_{-.5}^{.5} \sin^{-2} \pi u dF(u)$ is finite. Q.E.D. (1).

The distribution function $F(u)$ defines a Lebesgue-Stieltjes measure on the real line segment $-.5 \leqq u \leqq .5$. Let $\boldsymbol{W}$ denote the L^2 space of complex-valued measurable functions $\Phi(u)$ defined on $-.5 \leqq u \leqq .5$ for this measure. Define a correspondence between an element x of $\boldsymbol{X}$ and an element $\Phi(u)$ of $\boldsymbol{W}$ by

$$x = \int_{-.5}^{.5} \Phi(u) dx(0)E(u) \leftrightarrow \Phi(u).$$

Then Stone [5] and Kolmogorov [1] have shown that this correspondence establishes an isomorphism between $\boldsymbol{X}$ and $\boldsymbol{W}$ that preserves inner products. The function $e^{2\pi iut}/(1 - e^{2\pi iu})$ belongs to $\boldsymbol{W}$ since

$$\int_{-.5}^{.5} \left|\frac{e^{2\pi iut}}{1 - e^{2\pi iu}}\right|^2 dF(u) = \frac{1}{4}\int_{-.5}^{.5} \sin^{-2} \pi u du < \infty.$$

If we define the element $y(t)$ of $\boldsymbol{X}$ by the correspondence $y(t) \leftrightarrow e^{2\pi i u t}/(1-e^{2\pi i u})$ we see that

$$y(t) - y(t+1) \leftrightarrow \frac{e^{2\pi i u t} - e^{2\pi i u(t+1)}}{1 - e^{2\pi i u}} = e^{2\pi i u t}.$$

But by the spectral representation, we know that $x(t) \leftrightarrow e^{2\pi i u t}$, and hence we have $y(t)-y(t+1)=x(t)$ for all integers t. Since

$$\begin{aligned}(y(t+s), y(t)) &= \int_{-.5}^{.5} \frac{e^{2\pi i u(t+s)} e^{-2\pi i u t}}{|1 - e^{2\pi i u}|^2} dF(u) \\ &= \frac{1}{4} \int_{-.5}^{.5} e^{2\pi i u s} \sin^{-2} \pi u dF(u)\end{aligned}$$

depends only on s, we see that $y(t)$ is a stationary random process. Q.E.D. (2).

PROOF. (SUFFICIENT CONDITIONS). Let condition (1) of the theorem be given. Since

$$\text{variance} \left\{ \sum_1^n x(t) \right\} = \int_{-.5}^{.5} \frac{\sin^2 \pi u n}{\sin^2 \pi u} dF(u) \leqq \int_{-.5}^{.5} \sin^{-2} \pi u dF(u)$$

we see that the variance is bounded. Q.E.D. (1).

Let condition (2) of the theorem be given. Then $\|y(t)\|$ is a finite constant. Because $\sum_1^n x(t) = y(1) - y(n+1)$ we have $\|\sum_1^n x(t)\| \leqq \|y(1)\| + \|y(n+1)\|$, and so *variance* $\{\sum_1^n x(t)\} = \|\sum_1^n x(t)\|^2$ is bounded. Q.E.D. (2).

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MATHEMATICS RESEARCH CENTER, UNITED STATES ARMY,
MADISON, WISCONSIN