ON THE ABSOLUTE SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

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1. Let $f(x)$ be a function integrable in the Lebesgue sense and periodic with period 2π. Let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let its conjugate series be

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx).$$

We denote by $\sigma_n^\alpha(x)$ the nth Cesàro mean of order α of the Fourier series of $f(x)$. If the series

$$\sum_{n=1}^{\infty} | \sigma_n^\alpha(x) - \sigma_{n-1}^\alpha(x) |$$

converges, then we say that the Fourier series of $f(x)$ is absolutely summable (C, α) or summable $|C, \alpha|$ at the point x.

Further, we write

(i) $$W(\theta, t) = f(\theta + t) - f(\theta),$$

(ii) $$W_p(t) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(\theta + t) - f(\theta) |^p d\theta\right)^{1/p}.$$

The following two theorems were obtained by Chow [2, p. 440] as corollaries of theorems of Tsuchikura [3; 4] and Wang [5] respectively.

THEOREM A. *If*

$$W(\theta, t) = O\left\{\left(\log \frac{1}{|t|}\right)^{-(1+\delta)}\right\} \qquad (t \to 0)$$

uniformly in θ, *for some* $\delta > 0$, *then the Fourier series of* $f(x)$ *and its conjugate series are summable* $|C, \alpha|$ *everywhere for* $\alpha > 1/2$.

THEOREM B. *If*

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$$W_2(t) = O\left\{\left(\log\frac{1}{|t|}\right)^{-(1+\delta)}\right\} \qquad (t \to 0)$$

for some $\delta > 0$, *then the Fourier series of* $f(x)$ *and its conjugate series are summable* $|C, \alpha|$ *almost everywhere for* $\alpha > 1/2$.

As a generalization of Theorem B, Chow has established the following

THEOREM C [2]. *Let* $1 \leqq p \leqq 2$. *If*

$$W_p(t) = O\left\{\left(\log\frac{1}{|t|}\right)^{-(1+\delta)}\right\} \qquad (\delta > 0)$$

or, more generally, if

$$\text{(iii)} \qquad \int_{-\pi}^{\pi} \frac{W_p(t)}{|t|}\, dt < \infty,$$

then the Fourier series of $f(x)$ *and its conjugate series are both summable* $|C, \alpha|$ *almost everywhere for* $\alpha > 1/p$.

2. We write

$$\text{(iv)} \qquad \omega(\theta, t) = \int_0^t (f(\theta + u) - f(\theta))du,$$

$$\text{(v)} \qquad \Omega_p(t) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_0^t (f(\theta + u) - f(\theta))du\right|^p d\theta\right)^{1/p}.$$

We first prove that the condition

$$\text{(vi)} \qquad \int_{-\pi}^{\pi} W_p(t)dt < \infty$$

involves the boundedness of $\Omega_p(t)$. We have, by Minkowski's inequality, if $0 \leqq t \leqq \pi$,

$$\begin{aligned} \Omega_p(t) &\leqq \int_0^t \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta + u) - f(\theta)|^p d\theta\right)^{1/p} du \\ &= \int_0^t W_p(u)du \\ &\leqq \int_{-\pi}^{\pi} W_p(u)du \\ &< \infty. \end{aligned}$$

The same conclusion can be drawn for $-\pi \leqq t \leqq 0$ if we write

$$\Omega_p(t) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_t^0 (f(\theta+u)-f(\theta))du\right|^p d\theta\right)^{1/p}$$

and operate in a similar way.

Moreover, we can show that the condition (iii) implies, in fact, the following condition

$$\text{(vii)} \qquad \int_{-\pi}^{\pi} \frac{\Omega_p(t)}{t^2} dt < \infty.$$

We have, for $0 \leqq t \leqq \pi$,

$$\begin{aligned}\int_0^{\pi} \frac{\Omega_p(t)}{t^2} dt &\leqq \int_0^{\pi} \frac{dt}{t^2}\int_0^t W_p(u)du \\ &= \int_0^{\pi} W_p(u)du \int_u^{\pi} \frac{dt}{t^2} \quad \text{(by Fubini's theorem)} \\ &\leqq \int_0^{\pi} \frac{W_p(u)}{u} du \\ &\leqq \int_{-\pi}^{\pi} \frac{W_p(u)}{|u|} du \\ &< \infty.\end{aligned}$$

A similar argument gives the same conclusion for negative t.

We ask whether the condition (iii) of Chow's theorem can be replaced by the weaker conditions (vi) and (vii). Our answer is positive. In the present note, we improve Theorem C as follows:

THEOREM. *Let* $1 \leqq p \leqq 2$. *If* (vi) *and* (vii) *are satisfied, then the Fourier series of* $f(x)$ *and its conjugate series are summable* $|C, \alpha|$ *for* $\alpha > 1/p$.

3. The proof of the theorem is based on complex-variable methods. Let

$$c_0 = 1/2, \qquad c_n = a_n - ib_n \qquad (n \geqq 1).$$

Then the function

$$F(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}$$

is regular in the circle $|z| = r < 1$. The following lemma is due to Chow:

LEMMA 1. *The Fourier series of $f(x)$ and its conjugate series are summable $|C, \alpha|$ almost everywhere on the unit circle for $\alpha > 1/p$ provided that*

$$\int_0^r M_p(\rho, F')d\rho = \int_0^r \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} | F'(\rho e^{i\theta}) |^p d\theta\right)^{1/p} d\rho$$

is bounded as $r \to 1-0$.

Now,

$$\begin{aligned} F'(\rho e^{i\theta}) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)e^{it}}{(e^{it} - \rho e^{i\theta})^2} dt \\ &= \frac{e^{-i\theta}}{\pi}\int_{-\pi}^{\pi} \frac{f(\theta + t)e^{it}}{(e^{it} - \rho)^2} dt \\ &= \frac{e^{-i\theta}}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - \rho)^2} (f(\theta + t) - f(\theta))dt \quad [2, \text{p. } 441] \\ &= \frac{e^{i\theta}}{\pi}\left[\frac{e^{-it}}{(e^{it} - \rho)^2}\omega(\theta, t)\right]_{-\pi}^{\pi} - \frac{e^{-i\theta}}{\pi}\int_{-\pi}^{\pi} \omega(\theta, t) \frac{\partial}{\partial t}\left(\frac{e^{it}}{(e^{it} - \rho)^2}\right) dt \\ &= -\frac{1}{\pi}\frac{e^{-i\theta}}{(1 + \rho)^2}\omega(\theta, \pi) + \frac{1}{\pi}\frac{e^{-i\theta}}{(1 + \rho)^2}\omega(\theta, -\pi) \\ &\quad - \frac{e^{-i\theta}}{\pi}\int_{-\pi}^{\pi} \omega(\theta, t) \frac{\partial}{\partial t}\left(\frac{e^{it}}{(e^{it} - \rho)^2}\right) dt \\ &= \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta), \end{aligned}$$

say. We have

$$\begin{aligned} (2\pi)^{1/p}M_p(\rho, F') &= \left(\int_{-\pi}^{\pi} | F'(\rho e^{i\theta}) |^p d\theta\right)^{1/p} \\ &= \left(\int_{-\pi}^{\pi} | \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta) |^p d\theta\right)^{1/p} \\ &\leqq \left(\int_{-\pi}^{\pi} | \phi_1(\theta) |^p d\theta\right)^{1/p} + \left(\int_{-\pi}^{\pi} | \phi_2(\theta) |^p d\theta\right)^{1/p} \\ &\quad + \left(\int_{-\pi}^{\pi} | \phi_3(\theta) |^p d\theta\right)^{1/p} \end{aligned}$$

by Minkowski's inequality.

$$\left(\int_{-\pi}^{\pi} |\phi_1(\theta)|^p d\theta\right)^{1/p} \leqq \frac{1}{\pi} \frac{1}{(1+\rho)^2}\left(\int_{-\pi}^{\pi} |\omega(\theta, \pi)|^p d\theta\right)^{1/p}$$

$$= 2^{1/p}\pi^{1/p-1} \frac{\Omega_p(\pi)}{(1+\rho)^2}$$

$$= O\left(\frac{1}{(1+\rho)^2}\right).$$

A similar argument gives

$$\left(\int_{-\pi}^{\pi} |\phi_2(\theta)|^p d\theta\right)^{1/p} = O\left(\frac{1}{(1+\rho)^2}\right).$$

Finally, we have

$$\left(\int_{-\pi}^{\pi} |\phi_3(\theta)|^p d\theta\right)^{1/p}$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} \omega(\theta, t) \frac{\partial}{\partial t}\left(\frac{e^{it}}{(e^{it}-\rho)^2}\right) dt\right|^p d\theta\right)^{1/p}$$

$$\leqq \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\left|\omega(\theta, t) \frac{\partial}{\partial t}\left(\frac{e^{it}}{(e^{it}-\rho)^2}\right)\right|^p d\theta\right)^{1/p} dt$$

(by Minkowski's inequality)

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial t}\left(\frac{e^{it}}{(e^{it}-\rho^2)}\right)\right|\left(\int_{-\pi}^{\pi} |\omega(\theta, t)|^p d\theta\right)^{1/p} dt$$

$$= 2^{1/p}\pi^{1/p-1}\int_{-\pi}^{\pi} \Omega_p(t) \frac{1}{1-2\rho\cos t+\rho^2}\left(\frac{1+2\rho\cos t+\rho^2}{1-2\rho\cos t+\rho^2}\right)^{1/2} dt$$

$$= 2^{1/p}\pi^{1/p-1}\left(\int_{-\pi}^{0} + \int_{0}^{\pi}\right)$$

$$= 2^{1/p}\pi^{1/p-1}(I_1 + I_2).$$

We write, for $0<r<1$,

$$I_2 = \int_0^{1-r} + \int_{1-r}^{\pi/2} + \int_{\pi/2}^{\pi}$$

$$= I_3(\rho) + I_4(\rho) + I_5(\rho),$$

say, then

$$\int_0^r I_3(\rho)d\rho \leqq 2\int_0^r\left(\int_0^{1-r} \Omega_p(t)\,\frac{dt}{(1-\rho)^3}\right)d\rho$$
$$= 2\int_0^{1-r} \Omega_p(t)\left(\int_0^r \frac{d\rho}{(1-\rho)^3}\right)dt$$
$$\leqq \int_0^{1-r} \Omega_p(t)\,\frac{dt}{(1-r)^2}$$
$$\leqq \int_0^{1-r} \frac{\Omega_p(t)}{t^2}\,dt.$$

$$\int_0^r I_4(\rho)d\rho$$
$$= \int_0^r\left(\int_{1-r}^{\pi/2} \Omega_p(t)\,\frac{1}{1-2\rho\cos t+\rho^2}\left(\frac{1+2\rho\cos t+\rho^2}{1-2\rho\cos t+\rho^2}\right)\right)^{1/2} dt$$
$$\leqq 2\int_0^{\pi/2} \Omega_p(t)\left(\int_0^1 \frac{d\rho}{(1-2\rho\cos t+\rho^2)^{3/2}}\right)dt.$$

Now,

$$\int_0^1 \frac{d\rho}{(1-2\rho\cos t+\rho^2)^{3/2}} = \int_0^1 \frac{d\rho}{(\sin^2 t+(\rho-\cos t)^2)^{3/2}}$$
$$= \int_0^1 \frac{d(\rho-\cos t)}{(\sin^2 t+(\rho-\cos t)^2)^{3/2}} \qquad (t \text{ fixed})$$
$$= \frac{1}{\sin^2 t}\left(\sin\frac{t}{2}+\cos t\right)$$
$$\leqq 2/\sin^2 t$$
$$\leqq \pi^2/2t^2$$

for $0 \leqq t \leqq \pi/2$. It follows that

$$\int_0^r I_4(\rho)d\rho \leqq \pi^2\int_0^{\pi/2} \frac{\Omega_p(t)}{t^2}\,dt.$$

For $\pi/2 \leqq t \leqq \pi$, we have

$$\frac{1}{1-2\rho\cos t+\rho^2}\left(\frac{1+2\rho\cos t+\rho^2}{1-2\rho\cos t+\rho^2}\right)^{1/2} \leqq \frac{1}{1-2\rho\cos t+\rho^2}\,.$$

Thus,

$$\int_0^r I_5(\rho)d\rho \leqq \int_{\pi/2}^{\pi} \Omega_p(t)\left(\int_0^1 \frac{d\rho}{1-2\rho\cos t+\rho^2}\right)dt.$$

Since, for $\pi/2 \leqq t \leqq \pi$,

$$\begin{aligned}\int_0^1 \frac{d\rho}{1-2\rho\cos t+\rho^2} &= \frac{\pi - t}{2\sin t}\\ &\leqq \pi/4\\ &\leqq \frac{\pi^3}{4}\frac{1}{t^2}.\end{aligned}$$

Hence,

$$\int_0^r I_5(\rho)d\rho \leqq \frac{\pi^3}{4}\int_{\pi/2}^{\pi} \frac{\Omega_p(t)}{t^2}\,dt.$$

From the above analysis, we obtain

$$\int_0^r I_2(\rho)d\rho \leqq A\int_0^{\pi} \frac{\Omega_p(t)}{t^2}\,dt.$$

Similarly,

$$\int_0^r I_1(\rho)d\rho \leqq B\int_{-\pi}^{0} \frac{\Omega_p(t)}{t^2}\,dt.$$

Combining these two relations, we get finally

$$\int_0^r M_p(\rho, F')d\rho \leqq C\int_{-\pi}^{\pi} \frac{\Omega_p(t)}{t^2}\,dt + K\int_0^1 \frac{d\rho}{(1+\rho)^2},$$

where A, B, C and K are positive constants. Our theorem is thus completely established.

References

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