LIPSCHITZIAN PARAMETERIZATIONS AND EXISTENCE OF MINIMA IN THE CALCULUS OF VARIATIONS

GEORGE M. EWING

This note exhibits a brief and relatively elementary approach which the author has used when time and other exigencies precluded a more conventional development. The reader is referred to the recent paper of Cesari [1], particularly §§8, 10, 11.

Let x be a continuous rectifiable mapping of a closed interval $[a, b]$ into a fixed bounded closed subset A of the E_n and let $[t_{i-1}, t_i]$ denote a general subinterval of $[a, b]$ under a partition. Let f be a real function of $(x, r) \in A \times E_n$ subject to the conditions

I. f is continuous in (x, r),

II. $f(x, kr) = kf(x, r)$, $k \geqq 0$,

III. $f(x, r) > 0$, $r \neq 0$.

Letting the norm of the partition tend to 0, one then defines the Weierstrass integral denoted here by $W[x; a, b; f]$ as the limit,

$$\lim \sum f[x(T_i), x(t_i) - x(t_{i-1})]. \tag{1}$$

The limit exists independently of the choice of $T_i \in [t_{i-1}, t_i]$ and if $y(u)$, $u \in [c, d]$ is Fréchet-equivalent [1, p. 494] to $x(t)$, $t \in [a, b]$ then [2, p. 679],

$$W[y; c, d; f] = W[x; a, b; f]. \tag{2}$$

We require further of f that for each admissible x there exists on each subinterval $[t, t']$ of the parameter interval a number T such that

IV. $f[x(T), x(t') - x(t)] \leqq W[x; t, t'; f] + |x(t') - x(t)|^2$.

Mapping x is termed *f-Lipschitzian* (abbreviated fL) on $[a, b]$ if there is a constant k and on each subinterval $[t, t']$ a point T such that

$$f[x(T), x(t') - x(t)] \leqq k|t' - t|. \tag{3}$$

LEMMA 1. *A necessary and sufficient condition for x to be fL on $[a, b]$ is that x be Lipschitzian on $[a, b]$.*

PROOF. Using conditions I, III, observe that there exist positive constants m, M, such that for $x \in A$ and $|r| = 1$, $m \leqq f(x, r) \leqq M$. This holds for the unit vector $r/|r|$ when $r \neq 0$, while $f(x, 0) = 0$ from II. The stated result then follows from II and the above inequalities.

Received by the editors March 5, 1959 and, in revised form, March 23, 1959.

LEMMA 2. *A necessary and sufficient condition for x to be fL on $[a, b]$ is that there exist a constant k and on each subinterval $[t, t']$ of $[a, b]$ a number T such that*

$$f[x(T), x(t') - x(t)] \leqq k \mid t' - t \mid + \mid x(t') - x(t) \mid^2. \tag{4}$$

Clearly (3) implies (4). Given (4) let $[t, t']$ now be fixed and let π be a partition of $[t, t']$. Applying (4) separately to the subintervals, adding results, and considering limits as norm π tends to 0 with reference to (1), we find that $W[x; t, t'; f] \leqq k|t'-t|$. However, for a subinterval $[\tau, \tau']$ of $[t, t']$ under π, $m|x(\tau') - x(\tau)| \leqq f[x(T), x(\tau')-x(\tau)]$; hence from (1) applied both to the given f and to the function λ, $\lambda(x, r)=|r|$, it follows that $mW[x; t, t'; \lambda] \leqq W[x; t, t'; f]$. The integral on the left is simply the length of the mapping. It follows that $m|x(t')-x(t)| \leqq k|t'-t|$; hence that x is Lipschitzian with constant k/m.

Consider the class K of all admissible mappings x such that each $x \in K$ has the unit parameter-interval, satisfies (4) for some k, and is Fréchet-equivalent to each other $x \in K$.

LEMMA 3. *The set of numbers k associated with mappings x of K has a minimum, viz.* min $k = J(C, f)$ *denoting the common value of all integrals* (2) *for $x, y \in K$.*

PROOF. The equivalence class K includes the particular parameterization ξ in terms of *reduced J-length*, i.e. the mapping ξ such that for each subinterval $[t, t']$ of $[0, 1]$

$$W[\xi; t, t'; f] = (t' - t)J(C, f). \tag{5}$$

Existence of ξ can be established along the lines of [1, §10]. If one accepts the existence of at least one light parameterization, e.g. that in terms of reduced length $t/L(C)$, then ξ is obtained quickly from condition III on f and the nature of strictly increasing functions.

Since any mapping $x \in K$ satisfies (4) on the unit interval, we find by an argument used in the proof of Lemma 2 that $W[x; 0, 1; f] \leqq k$; hence that $J(C, f) \leqq k_0$, the infimum of values k for which (4) holds on the class K. Applying IV to the particular mapping ξ and using (5) we see that $J(C, f)$ is a particular k for which (4) holds on K. Thus $k_0 \leqq J(C, f)$ so that actually the equality holds and k_0 being realized through ξ is a minimum.

EXISTENCE THEOREM. *Let X denote the class of all admissible parameterizations x whose graphs join disjoint closed subsets of A. If X is nonempty and f has properties* I, II, III, IV *there exists $x_0 \in X$ minimizing W in X.*

PROOF. Denote the infimum of W on X by k_0. Let x_ν, $\nu=1, 2, \cdots$ be a sequence on X, which in the light of Lemma 3 can be chosen so that x_ν has the unit parameter interval and satisfies (4) with constant k_ν, $\lim k_\nu = k_0$. With the aid of Lemmas 1, 2, we see that the x_ν are all Lipschitzian with a common constant; hence that they are equicontinuous on $[0, 1]$ and by Ascoli's theorem [**3**, p. 336] we can suppose sequence x_ν to have been chosen so as to converge uniformly on $[0, 1]$ to a limit x_0. Given a subinterval $[t, t']$ of $[0, 1]$, then to each ν corresponds a number $T_\nu \in [t, t']$ such that (4) holds with x_ν, k_ν, T_ν. Thus a suitable subsequence of x_ν again denoted by x_ν has the property that T_ν converges to $T_0 \in [t, t']$. It follows that (4) holds for x_0, k_0, T_0 since otherwise (4) in x_ν, k_ν, T_ν is false for sufficiently large ν. It follows from (4) that $W[x_0; 0, 1; f] \leqq k_0$; hence by the definition of k_0 that equality must hold.

Certain types of side conditions could have been included in the definition of class X. The theorem can be rephrased in terms of Fréchet curves.

The writer has not been able to determine the relation between convexity of f in its second argument and the *regularity condition* IV. If $f(x, r) = \phi(x)g(r)$ with g convex in r then IV holds in *strict form*, i.e. without the second term on the right. If set A has an interior point b, if $\phi(x) = \text{constant}$, and if $g(r_1+r_2) > g(r_1)+g(r_2)$ then consideration of a short broken line issuing from b whose two segments have respective directions r_1, r_2, leads to a denial of the strict form of IV. However, the trivial subcase in which A is a segment and g is not convex does satisfy the strict form of IV.

Thus the class of problems covered by the theorem intersects nonvacuously with that included under theorems based on convexity and semi-continuity but is not included in the latter and probably vice versa.

Second terms on the right in conditions IV on f and (4) on x can be both replaced by any other function of the difference vector whose sum on subintervals of a partition tends to zero with the norm.

REFERENCES

1. L. Cesari, *Rectifiable curves and the Weierstrass integral*, Amer. Math. Monthly vol. 65 (1958) pp. 485–500.

2. G. M. Ewing, *Variation problems formulated in terms of the Weierstrass integral*, Duke Math. J. vol. 14 (1947) pp. 675–687.

3. E. J. McShane, *Integration*, Princeton University Press, 1944.

UNIVERSITY OF OKLAHOMA AND
THE U. S. ARMY ARTILLERY AND MISSILE SCHOOL, FORT SILL, OKLAHOMA