LIPSCHITZIAN PARAMETERIZATIONS AND EXISTENCE OF MINIMA IN THE CALCULUS OF VARIATIONS

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This note exhibits a brief and relatively elementary approach which the author has used when time and other exigencies precluded a more conventional development. The reader is referred to the recent paper of Cesari [1], particularly §§8, 10, 11.

Let $x$ be a continuous rectifiable mapping of a closed interval $[a, b]$ into a fixed bounded closed subset $A$ of the $E_n$ and let $[t_{i-1}, t_i]$ denote a general subinterval of $[a, b]$ under a partition. Let $f$ be a real function of $(x, r) \in A \times E_n$ subject to the conditions

I. $f$ is continuous in $(x, r)$,
II. $f(x, kr) = kf(x, r), k \geq 0$,
III. $f(x, r) > 0, r \neq 0$.

Letting the norm of the partition tend to 0, one then defines the Weierstrass integral denoted here by $W(x; a, b; f)$ as the limit,

$$
\lim \sum f(x(t_i), x(t_i) - x(t_{i-1})].
$$

The limit exists independently of the choice of $T_i \in [t_{i-1}, t_i]$ and if $y(u), u \in [c, d]$ is Fréchet-equivalent [1, p. 494] to $x(t), t \in [a, b]$ then [2, p. 679],

$$
W[y; c, d; f] = W[x; a, b; f].
$$

We require further of $f$ that for each admissible $x$ there exists on each subinterval $[t, t']$ of the parameter interval a number $T$ such that

IV. $f[x(T), x(t') - x(t)] \leq W[x; t, t'; f] + |x(t') - x(t)|^2$.

Mapping $x$ is termed $f$-Lipschitzian (abbreviated $fL$) on $[a, b]$ if there is a constant $k$ and on each subinterval $[t, t']$ a point $T$ such that

$$
f[x(T), x(t') - x(t)] \leq k |t' - t| .
$$

**Lemma 1.** A necessary and sufficient condition for $x$ to be $fL$ on $[a, b]$ is that $x$ be Lipschitzian on $[a, b]$.

**Proof.** Using conditions I, III, observe that there exist positive constants $m, M$, such that for $x \in A$ and $|r| = 1, m \leq f(x, r) \leq M$. This holds for the unit vector $r/|r|$ when $r \neq 0$, while $f(x, 0) = 0$ from II. The stated result then follows from II and the above inequalities.

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Lemma 2. A necessary and sufficient condition for \( x \) to be \( f \) on \([a, b]\) is that there exist a constant \( k \) and on each subinterval \([t, t']\) of \([a, b]\) a number \( T \) such that

\[
(4) \quad f(x(T), x(t') - x(t)) \leq k \left| t' - t \right| + \left| x(t') - x(t) \right|^2.
\]

Clearly (3) implies (4). Given (4) let \([t, t']\) now be fixed and let \( \pi \) be a partition of \([t, t']\). Applying (4) separately to the subintervals, adding results, and considering limits as norm \( \pi \) tends to 0 with reference to (1), we find that \( W[x; t, t'; f] \leq k \left| t' - t \right| \). However, for a subinterval \([\tau, \tau']\) of \([t, t']\) under \( \pi \), \( m \left| x(\tau') - x(\tau) \right| \leq f[x(T), x(t') - x(t)] \); hence from (1) applied both to the given \( f \) and to the function \( \lambda, \lambda(x, r) = |r| \), it follows that \( mW[x; t, t'; \lambda] \leq W[x; t, t'; f] \). The integral on the left is simply the length of the mapping. It follows that \( m \left| x(t') - x(t) \right| \leq k \left| t' - t \right| \); hence that \( x \) is Lipschitzian with constant \( k/m \).

Consider the class \( K \) of all admissible mappings \( x \) such that each \( x \in K \) has the unit parameter-interval, satisfies (4) for some \( k \), and is Fréchet-equivalent to each other \( x \in K \).

Lemma 3. The set of numbers \( k \) associated with mappings \( x \) of \( K \) has a minimum, viz. \( \min k = J(C, f) \) denoting the common value of all integrals (2) for \( x, y \in K \).

Proof. The equivalence class \( K \) includes the particular parameterization \( \xi \) in terms of reduced \( J \)-length, i.e. the mapping \( \xi \) such that for each subinterval \([t, t']\) of \([0, 1]\)

\[
(5) \quad W[\xi; t, t'; f] = (t' - t)J(C, f).
\]

Existence of \( \xi \) can be established along the lines of [1, §10]. If one accepts the existence of at least one light parameterization, e.g. that in terms of reduced length \( t/L(C) \), then \( \xi \) is obtained quickly from condition III on \( f \) and the nature of strictly increasing functions.

Since any mapping \( x \in K \) satisfies (4) on the unit interval, we find by an argument used in the proof of Lemma 2 that \( W[x; 0, 1; f] \leq k \); hence that \( J(C, f) \leq k_0 \), the infimum of values \( k \) for which (4) holds on the class \( K \). Applying IV to the particular mapping \( \xi \) and using (5) we see that \( J(C, f) \) is a particular \( k \) for which (4) holds on \( K \). Thus \( k_0 \leq J(C, f) \) so that actually the equality holds and \( k_0 \) being realized through \( \xi \) is a minimum.

Existence Theorem. Let \( X \) denote the class of all admissible parameterizations \( x \) whose graphs join disjoint closed subsets of \( A \). If \( X \) is nonempty and \( f \) has properties I, II, III, IV there exists \( x_0 \in X \) minimizing \( W \) in \( X \).
Proof. Denote the infimum of $W$ on $X$ by $k_0$. Let $x_\nu, \nu = 1, 2, \ldots$ be a sequence on $X$, which in the light of Lemma 3 can be chosen so that $x_\nu$ has the unit parameter interval and satisfies (4) with constant $k_\nu \lim k_\nu = k_0$. With the aid of Lemmas 1, 2, we see that the $x_\nu$ are all Lipschitzian with a common constant; hence that they are equi-
continuous on $[0, 1]$ and by Ascoli's theorem [3, p. 336] we can sup-
pose sequence $x_\nu$ to have been chosen so as to converge uniformly on $[0, 1]$ to a limit $x_0$. Given a subinterval $[t, t']$ of $[0, 1]$, then to each $\nu$ corresponds a number $T_\nu \in [t, t']$ such that (4) holds with $x_\nu, k_\nu, T_\nu$.
Thus a suitable subsequence of $x_\nu$ again denoted by $x_\nu$ has the prop-
erty that $T_\nu$ converges to $T_0 \in [t, t']$. It follows that (4) holds for $x_0, k_0, T_0$ since otherwise (4) in $x_\nu, k_\nu, T_\nu$ is false for sufficiently large $\nu$. It follows from (4) that $W[x_0; 0, 1; f] \leq k_0$; hence by the definition of $k_0$ that equality must hold.

Certain types of side conditions could have been included in the definition of class $X$. The theorem can be rephrased in terms of Fréchet curves.

The writer has not been able to determine the relation between convexity of $f$ in its second argument and the regularity condition IV. If $f(x, r) = \phi(x)g(r)$ with $g$ convex in $r$ then IV holds in strict form, i.e. without the second term on the right. If set $A$ has an interior point $b$, if $\phi(x) = \text{constant}$, and if $g(r_1 + r_2) > g(r_1) + g(r_2)$ then consideration of a short broken line issuing from $b$ whose two segments have respective directions $r_1, r_2$, leads to a denial of the strict form of IV. However, the trivial subcase in which $A$ is a segment and $g$ is not convex does satisfy the strict form of IV.

Thus the class of problems covered by the theorem intersects non-
vacuously with that included under theorems based on convexity and semi-continuity but is not included in the latter and probably vice versa.

Second terms on the right in conditions IV on $f$ and (4) on $x$ can be both replaced by any other function of the difference vector whose sum on subintervals of a partition tends to zero with the norm.

References


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