ON COMPLETE SEMIMODULES[1]

V. RAY HANCOCK

1. In the first theorem of [**3**], Wiegandt incorrectly states that a semimodule is complete if and only if it is divisible. (See §2 for definitions.) In this paper we show (in §3) that his theorem should read: a semimodule is complete if and only if it is a divisible group. Divisible semimodules which are not necessarily groups are discussed in §4, in connection with Wiegandt's second theorem. It is shown that a weakly integrally closed, divisible semimodule can be expressed as the direct sum of a divisible (abelian) group and a cone in a rational vector space.

The author wishes to acknowledge his gratitude to Professor A. H. Clifford for his many valuable comments.

2. Following Rédei [**1**] and Wiegandt [**2**; **3**], we shall use the following definitions.

A *semimodule* is a commutative, cancellative (Wiegandt: regular) semigroup with identity, which we shall write additively.

If Q, S, and T are semimodules, then T is called a *Schreier semimodule extension* of S by Q if S is a subsemimodule of T and for each $a \in Q$ there is a unique element u_a of T such that: (1) $u_0 = 0$, (2) the cosets $u_a + S$, $a \in Q$, form a partition of T, and (3) $u_a + u_b \in u_{a+b} + S$. Note that if on the set $\{u_a + S \mid a \in Q\}$, which we denote by T/S, we define the binary operation $\circ$ by $(u_a + S) \circ (u_b + S) = u_{a+b} + S$, then $T/S \cong Q$ under the mapping $u_a + S \to a$.

If T is a Schreier semimodule extension of S by Q then (3) implies the existence of a function $\phi: Q \times Q \to S$, called the factor-system of the extension, such that $u_a + u_b = u_{a+b} + \phi(a, b)$. This function ϕ satisfies the three laws:

$$\text{I.} \qquad \phi(a, 0) = 0,$$

$$\text{II.} \qquad \phi(a, b) = \phi(b, a),$$

$$\text{III.} \qquad \phi(a, b + c) + \phi(b, c) = \phi(a + b, c) + \phi(a, b),$$

for all a, b, c in Q. I reflects the condition $u_0 = 0$, II the commutativity of T, and III the associativity of T. If we define $+$ in $Q \times S$ by

$$\text{IV.} \qquad (a, \alpha) + (b, \beta) = (a + b, \phi(a, b) + \alpha + \beta),$$

Received by the editors April 6, 1959.

[1] This paper was prepared with the partial support of the National Science Foundation grant to Tulane University.

then $u_a+\alpha\to(a, \alpha)$ induces an isomorphism $T\cong Q\times S$. Conversely, if Q and S are semimodules and if there exists a function $\phi: Q\times Q\to S$ satisfying I, II, and III, then the semigroup T defined on $Q\times S$ by IV is a Schreier semimodule extension of S by Q with $u_a=(a, 0)$ and with each element α of S "identified" in the usual way with $(0, \alpha)$.

A semimodule is called *complete* if it is a direct summand of every Schreier semimodule extension of itself. A semimodule S is called *divisible* if for every $a\in S$ and every natural number n there exists $x\in S$ such that $nx=a$. If, moreover, x is uniquely determined by n and a, for every n and every a, then S is called *uniquely divisible.*

We shall use the following notations: If S is a semimodule, S^* shall denote the difference-group[2] of S and $U(S)$ shall denote the group of units[3] of S. We shall use N, N_0, R, and R_0 to denote, respectively, the sets of all natural numbers, all non-negative integers, all positive rationals, and all non-negative rationals.

A semimodule S is called *weakly integrally closed* if S contains every element α^* of S^* for which there exists n in N such that $n\alpha^*$ is in S. We shall use the term *cone* to signify a convex subset C of a rational vector space such that if $p\in C$ then $R_0p\subseteq C$ but $-Rp\cap C=\square$ (the empty set).

3. THEOREM 1. *If T is a Schreier semimodule extension of a semimodule S by a semimodule Q such that T^* is the direct sum $D^*\oplus S^*$ for some group D^*, and if D denotes the projection of T into D^*, then*:

(1) *D^* is the difference group of D (thereby justifying our use of the symbols D^* and D)*;

(2) *each element of D is in one of the cosets u_a+S^* $(a\in Q)$ and each such coset contains exactly one element of D*;

(3) *if $f: Q\to S^*$ is the function for which $v_a=u_a+f(a)$ is that unique element of D in u_a+S^*, then $f(0)=0$ and*

$$\phi(a, b) = f(a + b) - f(a) - f(b);$$

(4) *$T=D\oplus S$ if and only if $f(Q)\subseteq U(S)$.*

PROOF. We first observe that from the definition of D, clearly $T\subseteq D\oplus S^*$.

(1) If D' denotes the difference-group of D then $D\subseteq D^*$ and $T\subseteq D\oplus S^*$ imply

$$D' \subseteq D^* \subseteq D^* + S^* = T^* \subseteq (D \oplus S^*)^* = D' \oplus S^*,$$

[2] Every semimodule can be embedded in a group, unique to within isomorphism, the elements of which are each expressible as the difference of two elements of the semimodule.

[3] By a unit of a semimodule S we mean an element of S which has an inverse in S.

so that $D^* \subseteq D'$ since $D^* \cap S^* = \{0\}$. Thus $D^* = D'$.

(2) If $x \in D$, then there exist $t \in T$, $\alpha^* \in S^*$ such that $t = x + \alpha^*$, and there exist $a \in Q$, $\beta \in S$ such that $t = u_a + \beta$. Therefore we have (in T^*):

$$x = t - \alpha^* = u_a + \beta - \alpha^* \in u_a + S^*.$$

Conversely, since $u_a \in T \subseteq D \oplus S^*$, there exist $x \in D$, $\alpha^* \in S^*$ such that $u_a = x + \alpha^*$ and so $x = u_a - \alpha^* \in (u_a + S^*) \cap D$. If $x = u_a + \alpha^*$ and $y = u_a + \beta^*$ are both in $(u_a + S^*) \cap D$, then $u_a = x - \alpha^* = y - \beta^* \in D \oplus S^*$, whence $x = y$.

(3) From $v_a \in D \subseteq D^* \subseteq D^* \oplus S^*$ we have $v_a + v_b = v_{a+b}$, and so $u_{a+b} + \phi(a, b) = u_a + u_b = v_a + v_b - f(a) - f(b) = u_{a+b} + f(a+b) - f(a) - f(b)$. Thus $\phi(a, b) = f(a+b) - f(a) - f(b)$. Also, $D \cap S = \{0\}$ implies $v_0 = 0$; so that from $u_0 = 0$ we have $f(0) = 0$.

(4) If $f(Q) \subseteq U(S)$, then $v_a = u_a + f(a) \in T + S \subseteq T$ for every $a \in Q$, and hence $D \subseteq T$ and $D + S \subseteq T$. On the other hand, for each $t \in T$ there exist $a \in Q$, $\alpha \in S$ such that $t = u_a + \alpha = v_a - f(a) + \alpha \in D + S$; and hence $T \subseteq D + S$. From $D + S \subseteq D^* \oplus S^*$ we have $T = D + S = D \oplus S$. Conversely, if $T = D \oplus S$, then there exists a function $g: Q \to S$ such that $u_a = v_a + g(a)$ for each $a \in Q$. But then $u_a + 0 = u_a = v_a + g(a) = u_a + f(a) + g(a)$, so that $f(a) + g(a) = 0$. Thus $f(a) \in U(S)$ provided $f(a) \in S$. From $v_a \in T$, we have $v_a = u_b + \alpha$ for some $b \in Q$, $\alpha \in S$. However, the cosets $u_a + S^*$ and $u_b + S^*$ are either disjoint or identical. Since they both contain v_a, they must be identical and hence $u_a = u_b$ and $f(a) = \alpha \in S$. Q.E.D.

COROLLARY 2. *Any subgroup S of a semimodule T which is a direct summand of T^* must also be a direct summand of T.*

PROOF. If S is a group, then the cosets of S in T are either disjoint or identical (cf. Rédei [1]), and therefore form a partition of T for any choice of representatives. Thus T is a Schreier semimodule extension of S by some semimodule Q. By hypothesis, $T^* = D^* \oplus S^*$ for some group D^*, and for the corresponding function f,

$$f(Q) \subseteq S^* = S = U(S).$$

Hence, by Theorem 1, $T = D \oplus S$. Q.E.D.

COROLLARY 3. *An abelian group S is a direct summand of every semimodule T in which S is a subgroup if and only if S is divisible.*

PROOF. The necessity of divisibility is an immediate consequence of the corresponding group theorem. Conversely if S is a divisible subgroup of a semimodule T then S $(=S^*)$ must be a direct summand of T^*. We can therefore apply Corollary 2 to obtain the conclusion. Q.E.D.

THEOREM 4. *If T is a Schreier semimodule extension of a semimodule S by a semimodule Q with factor-system $\phi: Q\times Q\to S$, then S is a direct summand of T if and only if there exists a function $f: Q\to U(S)$ such that $f(0)=0$ and $\phi(a, b)=f(a+b)-f(a)-f(b)$ for all $a, b\in Q$. If S is a direct summand of T, then $\phi(Q\times Q)\subseteq U(S)$.*

PROOF. If S is a direct summand of T, say $T=D\oplus S$, then $T^* = D^*\oplus S^*$, and, by Theorem 1, the function f exists and has the properties specified.

Conversely, if the function f exists and has the properties specified, let $v_a=u_a+f(a)$ for each $a\in Q$, and let $D=\{v_a | a\in Q\}$. Then $v_a+v_b = u_a+u_b+f(a)+f(b)=u_{a+b}+\phi(a, b)+f(a)+f(b)=u_{a+b}+f(a+b)=v_{a+b} \in D$. Thus D is a subsemigroup of T. Each coset u_a+S contains exactly one element of D. Therefore $D\cap S=\{0\}$ since $v_0=u_0+f(0) =0$; and if $v_a+\alpha=v_b+\beta$ with $\alpha, \beta\in S$, then $a=b$ and $\alpha=\beta$. Thus $D+S=D\oplus S$. For each $t\in T$ there exist $a\in Q$, $\alpha\in S$ such that $t=u_a+\alpha=v_a-f(a)+\alpha\in D\oplus S$. That is, $T\subseteq D\oplus S$ and hence $T =D\oplus S$.

The final statement of the theorem is an immediate consequence of the existence of the function f satisfying $f(Q)\subseteq U(S)$ and $\phi(a, b) =f(a+b)-f(a)-f(b)$. Q.E.D.

The following is a corrected version of Wiegandt's Theorem 1 [3].

THEOREM 5. *A semimodule S is complete if and only if it is a divisible (abelian) group.*

PROOF. If S is a divisible (abelian) group then Corollary 3 assures us that it is a complete semimodule.

Conversely, if S is complete, then Wiegandt has proved that it must be divisible. Suppose S were not a group and let $\alpha\in S\setminus U(S)$ (i.e.: let α be a nonunit of S). Let T be the Schreier extension of S by N_0 with factor-system $\phi: N_0\times N_0\to S$ given by

$$\phi(a, b) = ab\alpha$$

where ab denotes the ordinary product of integers. Clearly ϕ satisfies I and II. To verify III we observe that $\phi(a, b+c)+\phi(b, c)=a(b+c)\alpha +bc\alpha = ab\alpha+ac\alpha+bc\alpha$, and $\phi(a+b, c) + \phi(a, b) = (a+b)c\alpha+ab\alpha =ac\alpha+bc\alpha+ab\alpha$. Thus T is a Schreier semimodule extension of S by N_0 for which $\phi(N_0\times N_0)\nsubseteq U(S)$. Hence by Theorem 4, S cannot be a direct summand of T, contradicting the completeness of S. Q.E.D.

4. Wiegandt's second theorem in [3] states that every complete semimodule is uniquely expressible as the direct sum of a divisible (abelian) group and a direct sum of semimodules each isomorphic to

the additive semimodule R_0. Our Theorem 5 shows that this theorem is correct, but superfluously stated. Since Wiegandt, basing his thinking on his Theorem 1, considered the terms *complete semimodule* and *divisible semimodule* to be synonymous, one is led naturally to the question: does Wiegandt's Theorem 2 remain valid when *complete* is replaced by *divisible*? A negative answer is provided by the following.

If a divisible semimodule S is decomposable as a direct sum $G \oplus P$ of a group G and a direct sum P of semimodules isomorphic to the additive semigroup R_0, then $G \subseteq U(S)$ and P is uniquely divisible. Consider, then, the multiplicative semimodule Z of all complex numbers z such that $0 < |z| < 1$, with 1 adjoined. Since $U(Z) = \{1\}$, the summand P, if it exists, would have to be all of Z, whereas Z is not uniquely divisible.

THEOREM 6. *If S is a divisible semimodule, then $U(S)$ is a direct summand of S.*

PROOF. As Wiegandt points out in his proof of his Theorem 2, $U(S)$ is a divisible abelian group. The conclusion is then an immediate consequence of Corollary 3. Q.E.D.

If S is a divisible semimodule, so that $S = U(S) \oplus C$ for some divisible semimodule C with $U(C) = \{0\}$, it does not follow that C^* is torsion-free, even though C itself cannot contain an element of finite order. And even if C^* were torsion-free, it is not necessarily true (as Wiegandt asserts) that C is a direct sum of its components in the decomposition of C^* into a direct sum of groups isomorphic to the additive rationals. However, we can prove the following.

THEOREM 7. (1) *A semimodule S is weakly integrally closed and divisible if and only if S is the direct sum of a divisible (abelian) group and a uniquely divisible semimodule with no nonzero units.*

(2) *A semimodule S with no nonzero units is uniquely divisible if and only if it is a cone in a rational vector space.*

PROOF. (1) If S is weakly integrally closed and divisible, then S^* is divisible; and since the identity 0 of S^* is the same as that of S, every element $\alpha^* \in S^*$ of finite order must be an element of S. Hence $U(S)$ will contain all the direct summands of S^* which are groups of Prüfer's type p^∞ for every p. By Theorem 6, $S = U(S) \oplus C$ for some subsemimodule C of S and clearly C must be a divisible subsemimodule of a direct sum of groups isomorphic to the additive group R_0^*. C is then necessarily uniquely divisible.

Conversely, the direct sum of a divisible abelian group G and a uniquely divisible semimodule P with no nonzero units will be a divisible semimodule. Suppose $\alpha^* \in (G \oplus P)^*$ is such that $n\alpha^* \in G \oplus P$. Then there exist $g \in G^*$, $p \in P^*$ such that $\alpha^* = g + p$, since $(G \oplus P)^* = G^* \oplus P^*$; and it follows that $ng \in G$ and $np \in P$. Now $g \in G = G^*$; and since P and P^* are both uniquely divisible, $np \in P$ and $p \in P^*$ imply $p \in P$. Thus $\alpha^* = g + p \in G \oplus P$ so that $G \oplus P$ is weakly integrally closed.

(2) If S is uniquely divisible, then S^* must be torsion-free and hence is a direct sum of groups isomorphic to the additive group R_0^*, i.e.: S^* is a rational vector space. By divisibility and semigroup closure, if $\alpha, \beta \in S$ then $R_0 \alpha \subseteq S$ and for each rational t with $0 \leqq t \leqq 1$, $t\alpha + (1-t)\beta \in S$. Moreover, if $\alpha \in S$ then, since $U(S) = \{0\}$, $-r\alpha$ cannot be in S for any $r \in R$. Thus S is a cone in S^*. Q.E.D.

In conclusion we note that every divisible semimodule S can be extended to its weak integral closure S' in S^*, defined by $S' = \{\alpha \in S^* \mid \exists n \in N \ni n\alpha \in S\}$, and the last theorem can then be applied to S' to give us some idea of the structure of S.

Bibliography

1. L. Rédei, *Die Verallgemeinerung der Schreierschen Erweiterungstheorie*, Acta Sci. Math. Szeged vol. 14 (1952) pp. 252–273.

2. R. Wiegandt, *On complete semi-groups*, Acta Sci. Math. Szeged vol. 19 (1958) pp. 93–97.

3. ———, *On complete semi-modules*, Acta Sci. Math. Szeged vol. 19 (1958) pp. 219–223.

Tulane University of Louisiana and
Virginia Polytechnic Institute