ON COMPLETE SEMIMODULES

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1. In the first theorem of [3], Wiegandt incorrectly states that a semimodule is complete if and only if it is divisible. (See §2 for definitions.) In this paper we show (in §3) that his theorem should read: a semimodule is complete if and only if it is a divisible group. Divisible semimodules which are not necessarily groups are discussed in §4, in connection with Wiegandt's second theorem. It is shown that a weakly integrally closed, divisible semimodule can be expressed as the direct sum of a divisible (abelian) group and a cone in a rational vector space.

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2. Following Rédei [1] and Wiegandt [2; 3], we shall use the following definitions.

A semimodule is a commutative, cancellative (Wiegandt: regular) semigroup with identity, which we shall write additively.

If Q, S, and T are semimodules, then T is called a Schreier semimodule extension of S by Q if S is a subsemimodule of T and for each a∈Q there is a unique element u_a of T such that: (1) u_0 = 0, (2) the cosets u_a+S, a∈Q, form a partition of T, and (3) u_a+u_b∈u_{a+b}+S. Note that if on the set \{u_a+S|a∈Q\}, which we denote by T/S, we define the binary operation o by (u_a+S) o (u_b+S) = u_{a+b}+S, then T/S≅Q under the mapping u_a+S→a.

If T is a Schreier semimodule extension of S by Q then (3) implies the existence of a function φ: Q×Q→S, called the factor-system of the extension, such that u_a+u_b = u_{a+b}+φ(a, b). This function φ satisfies the three laws:

I. \[ φ(a, 0) = 0, \]
II. \[ φ(a, b) = φ(b, a), \]
III. \[ φ(a, b + c) + φ(b, c) = φ(a + b, c) + φ(a, b), \]

for all a, b, c in Q. I reflects the condition u_0 = 0, II the commutativity of T, and III the associativity of T. If we define + in Q×S by

IV. \[ (a, α) + (b, β) = (a + b, φ(a, b) + α + β), \]

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then \( u_a + \alpha \rightarrow (a, \alpha) \) induces an isomorphism \( T \cong Q \times S \). Conversely, if \( Q \) and \( S \) are semimodules and if there exists a function \( \phi: Q \times Q \rightarrow S \) satisfying I, II, and III, then the semigroup \( T \) defined on \( Q \times S \) by IV is a Schreier semimodule extension of \( S \) by \( Q \) with \( u_a = (a, 0) \) and with each element \( \alpha \) of \( S \) "identified" in the usual way with \((0, \alpha)\).

A semimodule is called **complete** if it is a direct summand of every Schreier semimodule extension of itself. A semimodule \( S \) is called **divisible** if for every \( a \in S \) and every natural number \( n \) there exists \( x \in S \) such that \( nx = a \). If, moreover, \( x \) is uniquely determined by \( n \) and \( a \), for every \( n \) and every \( a \), then \( S \) is called **uniquely divisible**.

We shall use the following notations: If \( S \) is a semimodule, \( S^* \) shall denote the difference-group\(^2\) of \( S \) and \( U(S) \) shall denote the group of units\(^3\) of \( S \). We shall use \( N, N_0, R, \) and \( R_0 \) to denote, respectively, the sets of all natural numbers, all non-negative integers, all positive rationals, and all non-negative rationals.

A semimodule \( S \) is called **weakly integrally closed** if \( S \) contains every element \( \alpha^* \) of \( S^* \) for which there exists \( n \) in \( N \) such that \( n\alpha^* \) is in \( S \). We shall use the term **cone** to signify a convex subset \( C \) of a rational vector space such that if \( p \in C \) then \( R_0 p \subseteq C \) but \( -Rp \cap C = \emptyset \) (the empty set).

3. **Theorem 1.** If \( T \) is a Schreier semimodule extension of a semimodule \( S \) by a semimodule \( Q \) such that \( T^* \) is the direct sum \( D^* \oplus S^* \) for some group \( D^* \), and if \( D \) denotes the projection of \( T \) into \( D^* \), then:

1. \( D^* \) is the difference group of \( D \) (thereby justifying our use of the symbols \( D^* \) and \( D \));

2. each element of \( D \) is in one of the cosets \( u_a + S^* \) \((a \in Q)\) and each such coset contains exactly one element of \( D \);

3. if \( f: Q \rightarrow S^* \) is the function for which \( v_a = u_a + f(a) \) is that unique element of \( D \) in \( u_a + S^* \), then \( f(0) = 0 \) and

\[
\phi(a, b) = f(a + b) - f(a) - f(b);
\]

4. \( T = D \oplus S \) if and only if \( f(Q) \subseteq U(S) \).

**Proof.** We first observe that from the definition of \( D \), clearly \( T \subseteq D \oplus S^* \).

1. If \( D' \) denotes the difference-group of \( D \) then \( D \subseteq D^* \) and \( T \subseteq D \oplus S^* \) imply

\[
D' \subseteq D^* \subseteq D^* + S^* = T^* \subseteq (D \oplus S^*)^* = D' \oplus S^*;
\]

\(^2\) Every semimodule can be embedded in a group, unique to within isomorphism, the elements of which are each expressible as the difference of two elements of the semimodule.

\(^3\) By a unit of a semimodule \( S \) we mean an element of \( S \) which has an inverse in \( S \).
so that $D^* \subseteq D'$ since $D^* \cap S^* = \{0\}$. Thus $D^* = D'$.

(2) If $x \in D$, then there exist $t \in T$, $\alpha^* \in S^*$ such that $t = x + \alpha^*$, and there exist $a \in Q$, $\beta \in S$ such that $t = u_a + \beta$. Therefore we have (in $T^*$):

$$x = t - \alpha^* = u_a + \beta - \alpha^* \subseteq u_a + S^*.$$  

Conversely, since $u_a \in T \subseteq D \oplus S^*$, there exist $x \in D$, $\alpha^* \in S^*$ such that $u_a = x + \alpha^*$ and so $x = u_a - \alpha^* \in (u_a + S^*) \cap D$. If $x = u_a + \alpha^*$ and $y = u_a + \beta^*$ are both in $(u_a + S^*) \cap D$, then $u_a = x - \alpha^* = y - \beta^* \in D \oplus S^*$, whence $x = y$.

(3) From $v_a \in D \subseteq D^* \subseteq D^* \oplus S^*$ we have $v_a + v_b = v_{a+b}$, and so $u_a + \phi(a, b) = u_a + u_b = v_a + v_b - f(a) - f(b) = u_{a+b} + f(a+b) - f(a) - f(b)$. Thus $\phi(a, b) = f(a+b) - f(a) - f(b)$. Also, $D \cap S = \{0\}$ implies $v_0 = 0$; so that from $u_0 = 0$ we have $f(0) = 0$.

(4) If $f(Q) \subseteq U(S)$, then $v_a = u_a + f(a) \in T + S \subseteq T$ for every $a \in Q$, and hence $D \subseteq T$ and $D + S \subseteq T$. On the other hand, for each $t \in T$ there exist $a \in Q$, $\alpha \in S$ such that $t = u_a + \alpha = v_a - f(a) + \alpha \in D + S$; and hence $T \subseteq D + S$. From $D + S \subseteq D^* \oplus S^*$ we have $T = D + S = D \oplus S$. Conversely, if $T = D \oplus S$, then there exists a function $g: Q \rightarrow S$ such that $u_a = v_a + g(a)$ for each $a \in Q$. But then $u_a + 0 = u_a + v_a + g(a) = u_a + f(a) + g(a)$, so that $f(a) + g(a) = 0$. Thus $f(a) \in U(S)$ provided $f(a) \in S$. From $v_a \in T$, we have $v_a = u_a + \alpha$ for some $b \in Q$, $\alpha \in S$. However, the cosets $u_a + S^*$ and $u_b + S^*$ are either disjoint or identical. Since they both contain $v_a$, they must be identical and hence $u_a = u_b$ and $f(a) = \alpha \in S$. Q.E.D.

**Corollary 2.** Any subgroup $S$ of a semimodule $T$ which is a direct summand of $T^*$ must also be a direct summand of $T$.

**Proof.** If $S$ is a group, then the cosets of $S$ in $T$ are either disjoint or identical (cf. Rédei [1]), and therefore form a partition of $T$ for any choice of representatives. Thus $T$ is a Schreier semimodule extension of $S$ by some semimodule $Q$. By hypothesis, $T^* = D^* \oplus S^*$ for some group $D^*$, and for the corresponding function $f$,

$$f(Q) \subseteq S^* = S = U(S).$$

Hence, by Theorem 1, $T = D \oplus S$. Q.E.D.

**Corollary 3.** An abelian group $S$ is a direct summand of every semimodule $T$ in which $S$ is a subgroup if and only if $S$ is divisible.

**Proof.** The necessity of divisibility is an immediate consequence of the corresponding group theorem. Conversely if $S$ is a divisible subgroup of a semimodule $T$ then $S (= S^*)$ must be a direct summand of $T^*$. We can therefore apply Corollary 2 to obtain the conclusion. Q.E.D.
Theorem 4. If \( T \) is a Schreier semimodule extension of a semimodule \( S \) by a semimodule \( Q \) with factor-system \( \phi: \mathbb{Q} \times \mathbb{Q} \rightarrow S \), then \( S \) is a direct summand of \( T \) if and only if there exists a function \( f: \mathbb{Q} \rightarrow U(S) \) such that \( f(0) = 0 \) and \( \phi(a, b) = f(a+b) - f(a) - f(b) \) for all \( a, b \in \mathbb{Q} \). If \( S \) is a direct summand of \( T \), then \( \phi(\mathbb{Q} \times \mathbb{Q}) \subseteq U(S) \).

Proof. If \( S \) is a direct summand of \( T \), say \( T = D \oplus S \), then \( T^* = D^* \oplus S^* \), and, by Theorem 1, the function \( f \) exists and has the properties specified.

Conversely, if the function \( f \) exists and has the properties specified, let \( v_a = u_a + f(a) \) for each \( a \in \mathbb{Q} \), and let \( D = \{ v_a \mid a \in \mathbb{Q} \} \). Then \( v_a + v_b = u_a + u_b + f(a) + f(b) = u_{a+b} + \phi(a, b) + f(a) + f(b) = u_{a+b} + f(a+b) = v_{a+b} \in D \). Thus \( D \) is a subsemigroup of \( T \). Each coset \( u_a + S \) contains exactly one element of \( D \). Therefore \( D \cap S = \{ 0 \} \) since \( v_0 = u_0 + f(0) = 0 \); and if \( v_a + \alpha = v_b + \beta \) with \( \alpha, \beta \in S \), then \( a = b \) and \( \alpha = \beta \). Thus \( D + S = D \oplus S \). For each \( t \in T \) there exist \( a \in \mathbb{Q}, \alpha \in S \) such that \( t = u_a + \alpha = v_a - f(a) + \alpha \in D \oplus S \). That is, \( T \subseteq D \oplus S \) and hence \( T = D \oplus S \).

The final statement of the theorem is an immediate consequence of the existence of the function \( f \) satisfying \( f(\mathbb{Q}) \subseteq U(S) \) and \( \phi(a, b) = f(a+b) - f(a) - f(b) \). Q.E.D.

The following is a corrected version of Wiegandt's Theorem 1 [3].

Theorem 5. A semimodule \( S \) is complete if and only if it is a divisible (abelian) group.

Proof. If \( S \) is a divisible (abelian) group then Corollary 3 assures us that it is a complete semimodule.

Conversely, if \( S \) is complete, then Wiegandt has proved that it must be divisible. Suppose \( S \) were not a group and let \( \alpha \in S \setminus U(S) \) (i.e.: let \( \alpha \) be a nonunit of \( S \)). Let \( T \) be the Schreier extension of \( S \) by \( N_0 \) with factor-system \( \phi: N_0 \times N_0 \rightarrow S \) given by

\[
\phi(a, b) = ab\alpha
\]

where \( ab \) denotes the ordinary product of integers. Clearly \( \phi \) satisfies I and II. To verify III we observe that \( \phi(a, b+c) + \phi(b, c) = a(b+c)\alpha + bc\alpha = ab\alpha + ac\alpha + bc\alpha \), and \( \phi(a+b, c) + \phi(a, b) = (a+b)c\alpha + ab\alpha = ac\alpha + bc\alpha + ab\alpha \). Thus \( T \) is a Schreier semimodule extension of \( S \) by \( N_0 \) for which \( \phi(N_0 \times N_0) \subseteq U(S) \). Hence by Theorem 4, \( S \) cannot be a direct summand of \( T \), contradicting the completeness of \( S \). Q.E.D.

4. Wiegandt's second theorem in [3] states that every complete semimodule is uniquely expressible as the direct sum of a divisible (abelian) group and a direct sum of semimodules each isomorphic to
the additive semimodule $R_0$. Our Theorem 5 shows that this theorem is correct, but superfluously stated. Since Wiegandt, basing his thinking on his Theorem 1, considered the terms complete semimodule and divisible semimodule to be synonymous, one is led naturally to the question: does Wiegandt’s Theorem 2 remain valid when complete is replaced by divisible? A negative answer is provided by the following.

If a divisible semimodule $S$ is decomposable as a direct sum $G \oplus P$ of a group $G$ and a direct sum $P$ of semimodules isomorphic to the additive semigroup $R_0$, then $G \subseteq U(S)$ and $P$ is uniquely divisible. Consider, then, the multiplicative semimodule $Z$ of all complex numbers $z$ such that $0 < |z| < 1$, with 1 adjoined. Since $U(Z) = \{1\}$, the summand $P$, if it exists, would have to be all of $Z$, whereas $Z$ is not uniquely divisible.

**Theorem 6.** If $S$ is a divisible semimodule, then $U(S)$ is a direct summand of $S$.

**Proof.** As Wiegandt points out in his proof of his Theorem 2, $U(S)$ is a divisible abelian group. The conclusion is then an immediate consequence of Corollary 3. Q.E.D.

If $S$ is a divisible semimodule, so that $S = U(S) \oplus C$ for some divisible semimodule $C$ with $U(C) = \{0\}$, it does not follow that $C^*$ is torsion-free, even though $C$ itself cannot contain an element of finite order. And even if $C^*$ were torsion-free, it is not necessarily true (as Wiegandt asserts) that $C$ is a direct sum of its components in the decomposition of $C^*$ into a direct sum of groups isomorphic to the additive rationals. However, we can prove the following.

**Theorem 7.** (1) A semimodule $S$ is weakly integrally closed and divisible if and only if $S$ is the direct sum of a divisible (abelian) group and a uniquely divisible semimodule with no nonzero units.

(2) A semimodule $S$ with no nonzero units is uniquely divisible if and only if it is a cone in a rational vector space.

**Proof.** (1) If $S$ is weakly integrally closed and divisible, then $S^*$ is divisible; and since the identity 0 of $S^*$ is the same as that of $S$, every element $a^* \in S^*$ of finite order must be an element of $S$. Hence $U(S)$ will contain all the direct summands of $S^*$ which are groups of Prüfer's type $p^\infty$ for every $p$. By Theorem 6, $S = U(S) \oplus C$ for some subsemimodule $C$ of $S$ and clearly $C$ must be a divisible subsemimodule of a direct sum of groups isomorphic to the additive group $R_0^*$. $C$ is then necessarily uniquely divisible.

Conversely, the direct sum of a divisible abelian group $G$ and a uniquely divisible semimodule $P$ with no nonzero units will be a
divisible semimodule. Suppose $a^* \in (G \oplus P)^*$ is such that $na^* \in G \oplus P$. Then there exist $g \in G^*$, $p \in P^*$ such that $a^* = g + p$, since $(G \oplus P)^* = G^* \oplus P^*$; and it follows that $ng \in G$ and $np \in P$. Now $g \in G \subseteq G^*$; and since $P$ and $P^*$ are both uniquely divisible, $np \in P$ and $p \in P^*$ imply $p \in P$. Thus $a^* = g + p \in G \oplus P$ so that $G \oplus P$ is weakly integrally closed.

(2) If $S$ is uniquely divisible, then $S^*$ must be torsion-free and hence is a direct sum of groups isomorphic to the additive group $R_0^*$, i.e.: $S^*$ is a rational vector space. By divisibility and semigroup closure, if $\alpha, \beta \in S$ then $R_0 \alpha \subseteq S$ and for each rational $t$ with $0 \leq t \leq 1$, $t \alpha + (1-t) \beta \in S$. Moreover, if $\alpha \in S$ then, since $U(S) = \{0\}$, $-r \alpha$ cannot be in $S$ for any $r \in R$. Thus $S$ is a cone in $S^*$. Q.E.D.

In conclusion we note that every divisible semimodule $S$ can be extended to its weak integral closure $S'$ in $S^*$, defined by $S' = \{ \alpha \in S^* | \exists n \in \mathbb{N} \exists \alpha \in S \}$, and the last theorem can then be applied to $S'$ to give us some idea of the structure of $S$.

**Bibliography**


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