COMPACT LINEAR TRANSFORMATIONS

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1. Consider a compact linear transformation T (also called completely continuous transformation) from a Banach space A to a Banach space B. Can T be approximated arbitrarily close in norm by bounded linear transformations whose ranges are finite dimensional (see [**1**, p. 49])? The answer is affirmative for the following types of domain and range spaces: (i) both A and B are Hilbert spaces (see [**2**, p. 204]), (ii) both A and B are $C[a, b]$ (see [**2**, p. 222] or [**3**]), (iii) there is no other restriction on A, but B is of "type A" [**4**], (iv) A is either L^p or C and there is no other restriction on B (see [**5**, p. 536]). In this paper we shall show that the answer is also affirmative when A is any Banach space and B is $C(E)$, E being a compact Hausdorff space.

Let S^* be the strongly closed unit sphere in the conjugate space B^*, namely the set of all linear functionals of unit norm or less. S^* is a compact Hausdorff space in the relative topology introduced in S^* by the weak * topology of B^* (see [**1**, p. 37]). For convenience, we continue to call this relative topology in S^* the weak * topology. Denote by $C(S^*)$ the Banach algebra of all the complex-valued weak *-continuous functions in S^*. For each x in B, the mapping $x \to x^{**}$ is an isometric isomorphism embedding B as a subspace of B^{**}, and $x^{**} \to x^{**}$ (restricted to S^*) is also an isomorphism satisfying

$$\|x^{**}\| = \operatorname*{l.u.b.}_{F \in S^*} |x^{**}(F)| = \|x^{**}\|_\infty;$$

where $\|x^{**}\|_\infty$ is the uniform norm of x^{**} restricted to S^*. Hence we can embed B as a subspace of $C(S^*)$ under the isometric isomorphism $x \to x^{**}$ (restricted to S^*). Consequently, by embedding B in $C(S^*)$, a compact linear transformation T from a Banach space A to a Banach space B can be approximated arbitrarily close in norm by bounded linear transformations of finite dimensional range from A to $C(S^*)$. (See §3.)

The ideas in §§2 and 3 are suggested by those of Radon in [**3**]; (see also [**2**, p. 222]). Throughout this note, $C(E)$ denotes the Banach algebra of all complex-valued continuous functions defined on a compact Hausdorff space E, and $(T^*F)(x)$ means $(Tx)(F)$, for x in A, Tx in $C(E)$ and F in E.

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2. In this section we shall establish the following result:

Let T be a compact linear transformation from a Banach space A to the space $C(E)$. Then T^* is a continuous mapping on E to A^*. (Here A^* is given the usual norm topology.)

PROOF. Take a fixed F_0 in E. To each $\epsilon > 0$, we have to show that there is an open set O in E containing F_0 such that

$$\|T^*F - T^*F_0\| < \epsilon \qquad \text{for all } F \in O. \tag{1}$$

Suppose that this is not true. Then there is an $\epsilon = 2\epsilon_0 > 0$ such that (1) is satisfied by no open set O in E containing F_0. We show that it leads to contradiction.

By virtue of the compactness of T, the image $T(S)$ of the closed unit sphere S in A is separable and hence contains a sequence $\{z_n\}$ dense in the closure of $T(S)$.

The sets $U_{m,n} = \{F \mid |z_m(F) - z_m(F_0)| < 1/n,\ F \in E\}$ are open sets in E and form a sequence $\{V_k\}$. Let O_n be the intersection of V_1, $V_2, \cdots$ and V_n. Clearly F_0 lies in each O_n and $\{O_n\}$ is monotonic decreasing. By the supposition, for each n, there exists a F_n in O_n such that $\|T^*F_n - T^*F_0\| \geqq 2\epsilon_0$. But then there is a x_n in S such that

$$|[Tx_n](F_n) - [Tx_n](F_0)| = |[T^*F_n - T^*F_0](x_n)| > \epsilon_0. \tag{2}$$

Since T is compact, we may suppose, by passing to a subsequence if necessary, that

$$Tx_n = y_n \text{ converges in norm to some } y \text{ in } C(E). \tag{3}$$

As y can be approximated arbitrarily close in norm by $\{z_n\}$, to an integer p satisfying $3 < \epsilon_0 p$, there is an integer q such that

$$\|y - z_q\|_\infty < \frac{1}{p} < \frac{\epsilon_0}{3} \cdot \tag{4}$$

Let N be so large that O_N is contained in $U_{q,p}$. Then by (4), $|y(F_n) - z_q(F_n)| < 1/p$, $|y(F_0) - z_q(F_0)| < 1/p$ and, for $n \geqq N$, $|z_q(F_n) - z_q(F_0)| < 1/p$. Hence

$$|y(F_n) - y(F_0)| < 3/p < \epsilon_0 \qquad \text{for } n \geqq N. \tag{5}$$

Now that, by virtue of (3), $|y_n(F_n) - y(F_n)|$ and $|y_n(F_0) - y(F_0)|$ both tend to zero as $n \to \infty$ and (5) together show that (2) cannot hold for all n. The contradiction proves that T^* is continuous on E.

3. Let T be the transformations of §2. For each F_k in E and each $\epsilon > 0$, the set $O_k = \{F \mid \|T^*F - T^*F_k\| < \epsilon,\ F \in E\}$ is open. Let g_k be a real-valued continuous function in E such that $g_k = 2$ at F_k, $g_k = 0$

outside O_k, $0 \leqq g_k \leqq 2$. The existence of such functions is assured by the Urysohn's lemma. Set $U_k = \{F | g_k(F) > 1, F \text{ in } E\}$, then U_k is an open set containing F_k, and $U_k \subset O_k$. Since E is compact, it can be covered by some finite family of sets $U_1, U_2, \cdots, U_n$. Setting $h_1(F) = \inf(g_1(F), 1)$, we define inductively

$$h_m = \inf\left(\sum_{i=1}^{m-1} h_i + g_m, 1\right) - \sum_{i=1}^{m-1} h_i, \qquad m = 2, 3, \cdots, n.$$

The functions h_m are continuous and belong to $C(E)$. They satisfy

$$o \leqq h_i(F) \leqq 1, \qquad \sum_{i=1}^{n} h_i(F) = 1 \text{ in } E,$$

$h_i(F) \neq 0$ implies $F \in O_i$. For x in A, define

$$T_n x = \sum_{i=1}^{n} (Tx)(F_i) h_i.$$

Clearly $T_n x$ is in $C(E)$ and the range of T_n is finite dimensional. Using the properties of the functions h_i and the definition of O_i, we can see that in E

$$\begin{aligned} |(Tx)(F) - (T_n x)(F)| &= \left|(T^*F)(x) - \sum_{i=1}^{n} h_i(F)(T^*F_i)(x)\right| \\ &\leqq \left\|T^*F - \sum_{i=1}^{n} h_i(F) T^*F_i\right\| \|x\| \\ &\leqq \sum_{i=1}^{n} h_i(F) \|T^*F - T^*F_i\| \|x\| \\ &< \epsilon \|x\|. \end{aligned}$$

Hence

$$\|Tx - T_n x\|_\infty < \epsilon \|x\|, \qquad \|T - T_n\| \leqq \epsilon.$$

We have thus proved the following result:

A compact linear transformation T from a Banach space A to the space $C(E)$ can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range.

In view of the discussion in §1, it follows that

A compact linear transformation T from a Banach space A to a Banach space B, embedded in $C(S^*)$, can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from A to $C(S^*)$.

4. If a sequence of compact linear transformations converges to a limit in norm it is known that the limit is compact. (See [1, p. 49]). In view of this property, the first result in §3 can be stated as follows:

T is a compact linear transformation from a Banach space A to a Banach algebra $C(E)$ if and only if T can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from A to $C(E)$.

The method in §3 uses essentially the continuity of T^*. Hence from §§2 and 3 we see that

A bounded linear transformation T from a Banach space A to a Banach algebra $C(E)$ is compact if and only if T^* is continuous from E to A^*.

As consequences of these remarks we also see that

A linear transformation from a Banach space A to a Banach space B is compact if and only if when B is embedded in $C(S^*)$ it can be approximated arbitrarily close in norm by bounded linear transformations of finite-dimensional range from A to $C(S^*)$; and

A bounded linear transformation T from a Banach space A to a Banach space B is compact if and only if T^* is weak *-continuous on S^* to A^*.

Let $\beta(A, B)(\beta(A, C(S^*)))$ be the Banach space of all compact linear transformations from the Banach space A to the Banach space $B(C(S^*))$. We can also express the above results in the following form:

$\beta(A, B)$ can be embedded in $\beta(A, C(S^*))$. The subspace of all the transformations of finite-dimensional range in $\beta(A, C(S^*))$ is dense in $\beta(A, C(S^*))$.

When $A=B$, $\beta(A, A)$ is an algebra. To apply the results above, we can embed both the domain A and range A in $C(S^*)$.

We observe that the completeness of A has not been used in this note.

References

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