

HAUSDORFF-ANALYTIC FUNCTIONS OF MATRICES¹

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1. **Introduction.** The concept of Hausdorff-analytic (H -analytic) functions on an (associative) hypercomplex system, over the complex number field, with an identity [1] is, in this paper, applied to functions of matrices with distinct eigenvalues. In the process of showing a sufficient condition for a matrix function to be H -analytic in a neighborhood of a matrix Z_0 with distinct eigenvalues, it is shown that the projections (Frobenius covariants) are H -analytic in a neighborhood of Z_0 , and that there exists a matrix function $Q = Q(Z)$ which is H -analytic in a neighborhood of Z_0 such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_k)$, $\lambda_i \neq \lambda_j$ for $i \neq j$, for all Z in that neighborhood of Z_0 .

It has previously been shown [2] that if $f(Z)$ is a function on \mathfrak{M} (the algebra of all square matrices of order n over the complex number field) whose component functions are analytic functions, in some open domain, of the complex (component) variables z_{ij} of $Z = (z_{ij})$, then $f(Z)$ is H -analytic in a corresponding open domain of \mathfrak{M} .

2. H -analyticity of the projections of a matrix with distinct eigenvalues.

LEMMA 2.1. *The projections (or Frobenius covariants) $P_k(Z)$ corresponding to the eigenvalues λ_k , $k = 1, \dots, s$, of a matrix Z in \mathfrak{M} are given by*

$$P_k(Z) = \frac{1}{2\pi i} \int_{C_k} (\lambda I - Z)^{-1} d\lambda,$$

where C_k is a circle in the complex λ -plane containing λ_k but none of the other λ_j .

PROOF. By [4, p. 22],

$$(\lambda I - Z)^{-1} = \sum_{j=1}^s P_j(Z) \sum_{m=0}^{r_j-1} \frac{(Z - \lambda_j I)^m}{(\lambda - \lambda_j)^{m+1}},$$

where r_j is the index of λ_j .

The integral, over a curve in the complex λ -plane, of a matrix of complex functions is given by the matrix of integrals, over the curve, of the matrix elements. Therefore if we let C_k be a circle containing

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λ_k but none of the other λ_j , then

$$\begin{aligned} & \frac{1}{2\pi i} \int_{C_k} (\lambda I - Z)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \sum_{j=1}^s P_j(Z) \sum_{m=0}^{r_j-1} (Z - \lambda_j I)^m \int_{C_k} \frac{d\lambda}{(\lambda - \lambda_j)^{m+1}} = P_k(Z). \end{aligned}$$

We will use this representation to prove the following.

THEOREM 2.1. *If Z_0 is a matrix with distinct eigenvalues, then there exists a neighborhood N of Z_0 such that for Z in N , $P_k(Z)$ is an H -analytic function of Z .*

PROOF. Let $\lambda_k^0, k=1, \dots, n$, be the (distinct) eigenvalues of Z_0 , then

$$P_k(Z_0) = \frac{1}{2\pi i} \int_{C_k} (\lambda I - Z)^{-1} d\lambda$$

is the projection of Z_0 corresponding to λ_k^0 , where C_k is a sufficiently small circle which has λ_k^0 as its center and all other λ_j^0 in its exterior, that is, if $|\lambda_k^0 - \lambda_j^0| > 2\epsilon$ for $j \neq k$, then $C_k: |\lambda - \lambda_k^0| = \epsilon$ will be sufficient.

Now, for all matrices Z sufficiently near Z_0 , that is, such that $\text{norm}(Z - Z_0)$ is sufficiently small (where, for convenience, the norm of any matrix $X = (x_{ij}), i=1, \dots, m, j=1, \dots, n$, with complex components, shall be defined by $\text{norm}(X) = \max_{i,j} |x_{ij}|$), the eigenvalues $\lambda_1, \dots, \lambda_n$ of Z will be near those of Z_0 , since the eigenvalues of a matrix are continuous functions of the elements of the matrix [2]. Thus, a neighborhood N of Z_0 may be chosen such that $|\lambda_k - \lambda_k^0| < \epsilon, k=1, \dots, n$, then for each Z in N , Z has distinct eigenvalues λ_j , and λ_k lies within C_k while all other λ_j lie outside C_k . Hence for all Z in N ,

$$P_k(Z) = \frac{1}{2\pi i} \int_{C_k} (\lambda I - Z)^{-1} d\lambda.$$

The r, s element of the matrix $P_k(Z)$ is given by

$$P_k(Z)_{rs} = \frac{1}{2\pi i} \int_{C_k} R_{rs}(\lambda, z_{ij}) d\lambda,$$

where $R_{rs}(\lambda, z_{ij})$ is the quotient of two polynomials in λ and the $z_{ij}, i, j=1, \dots, n$. Since C_k does not pass through any of the zeros of $\det(\lambda I - Z)$, regardless of what Z in N is chosen, $R_{rs}(\lambda, z_{ij})$ is a continuous function of the complex variables λ and $z_{ij}, i, j=1, \dots, n$,

where each z_{ij} ranges over a region N_{ij} determined by N , and λ lies on C_k ; also $R_{rs}(\lambda, z_{ij})$ is an analytic function of each z_{ij} in N_{ij} , for every value of λ of C_k . Therefore, $P_k(Z)_{rs}$ is an analytic function of each z_{ij} of Z in N , and hence, the components $P_k(Z)_{rs}$ are analytic functions of z_{ij} , that is $P_k(Z)$ is H -analytic at Z in N .

3. The existence of $Q(Z)$.

LEMMA 3.1. *If Z_0 is a matrix with distinct eigenvalues and x is a vector such that $P_k(Z_0)x \neq 0$ for every $k=1, \dots, n$, then for Z sufficiently near Z_0 , $P_k(Z)x \neq 0$, $k=1, \dots, n$.*

PROOF. Since $P_k(Z_0)x \neq 0$ for every $k=1, \dots, n$, there exists a $\delta > 0$ such that $\text{norm}(P_k(Z_0)x) > \delta$ for every k . Now $P_k(Z)$ is an H -analytic function of Z in a neighborhood of Z_0 and therefore a continuous function of Z in a neighborhood of Z_0 . Thus, for Z near Z_0 , $P_k(Z)$ is near $P_k(Z_0)$ and therefore $P_k(Z)x$ is near $P_k(Z_0)x$; in particular Z may be chosen sufficiently close to Z_0 such that

$$\text{norm}(P_k(Z_0)x - P_k(Z)x) < \delta/2,$$

for all k . Hence, for all Z in such a neighborhood N of Z_0 ,

$$\begin{aligned} \text{norm}(P_k(Z)x) &\geq \text{norm}(P_k(Z_0)x) - \text{norm}(P_k(Z_0)x - P_k(Z)x) \\ &> \delta - \delta/2 = \delta/2 > 0. \end{aligned}$$

Thus, for Z in N , $P_k(Z)x \neq 0$ for every $k=1, \dots, n$.

THEOREM 3.1. *Let Z_0 be a matrix with distinct eigenvalues; then there exists a nonsingular matrix Q whose components are analytic functions of the elements of Z , for Z in some neighborhood N of Z_0 (therefore Q and Q^{-1} are H -analytic functions of Z in N), such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_k)$ for all Z in N .*

PROOF. Choose a vector x such that $P_k(Z_0)x \neq 0$ for all $k=1, \dots, n$, then by Lemma 3.1, for Z sufficiently near Z_0 , $Q_k = P_k(Z)x \neq 0$. By [4, p. 22], $(Z - \lambda_k I)Q_k = 0$, since the index r_k of λ_k is 1 for all k , that is, $ZQ_k = \lambda_k Q_k$. Now, let Q be the matrix whose j th column is Q_j , then $ZQ = Q\Lambda$, where $\Lambda = \text{diag}(\lambda_k)$. Therefore, since the Q_k are linearly independent $Q^{-1}ZQ = \Lambda$.

Since $Q_k = P_k(Z)x$, the components of Q_k are linear combinations of the elements of $P_k(Z)$ and therefore, by Theorem 2.1, they are analytic functions of the elements z_{ij} of Z in a neighborhood of Z_0 . Hence Q is H -analytic in a sufficiently small neighborhood N of Z_0 .

Also $Q^{-1} = (S_{ij}/\det(Q))$, where S_{ij} is the cofactor of the j, i component of Q . Since $\det(Q) \neq 0$, and S_{ij} and $\det(Q)$ are polynomials

with constant coefficients in the analytic components of Q , the components at Q^{-1} are analytic functions of the elements of Z in N and therefore Q^{-1} is also H -analytic in N .

Note: If one knows a Q_0 such that $Q_0^{-1}Z_0Q_0 = \Lambda_0 = \text{diag}(\lambda_k^0)$ for the given Z_0 , then, for the above x , one may choose $x = \sum_{j=1}^n Q_0^{(j)}$, where $Q_0^{(j)}$ is the j th column of Q_0 , since $Z_0Q_0^{(k)} = \lambda_k^0Q_0^{(k)}$, that is $P_k(Z_0)Q_0^{(j)} = \delta_{jk}Q_0^{(k)}$, and therefore $P_k(Z_0)x = Q_0^{(k)} \neq 0$ for every $k = 1, \dots, n$.

COROLLARY 3.1. *If the matrix Z has distinct eigenvalues $\lambda_k, k = 1, \dots, n$, then the λ_k are analytic functions of the components z_{ij} of Z .*

4. A sufficient condition for H -analyticity of a matrix function at a matrix with distinct eigenvalues. The hypothesis of (i) of the following theorem would be desirable if one wished to view this as a theory for functions of linear transformations of a finite dimensional vector space, for the invariance under similarity transformations, $F(Y) = P^{-1}F(X)P$ for $Y = P^{-1}XP$, permits the definition of a function of a finite dimensional linear transformation to be independent of the choice of basis for the vector space.

THEOREM 4.1. *Let $F = \sum_{i,j=1}^n f_{ij}E_{ij}$ be a matrix function (where E_{ij} is the $n \times n$ matrix with a 1 in the i, j position and zeros elsewhere).*

(i) *Let F be such that, F defined at X and $Y = P^{-1}XP$ implies that F is defined at Y and $F(Y) = P^{-1}F(X)P$; then, for a given X at which F is defined, $F(X)$ is a polynomial in X . In particular, if $\Lambda = \text{diag}(\lambda_i)$ is a diagonal matrix at which F is defined, then $F(\Lambda)$ is a diagonal matrix, that is,*

$$f_{ij} \Big|_{z_{rs} = \delta_{rs}\lambda_r} = \delta_{ij}g_i(\lambda_1, \dots, \lambda_n).$$

(ii) *Further, if F is also such that the diagonal component functions f_{ii} are analytic functions of each λ_j at the components of a diagonal matrix with distant eigenvalues λ_i , at which F is defined, that is,*

$$\left. \frac{\partial f_{ii}}{\partial z_{jj}} \right|_{z_{rs} = \delta_{rs}\lambda_r} = \frac{\partial g_i}{\partial \lambda_j}$$

exist for $i, j = 1, \dots, n$, then, F is H -analytic at a matrix Z with distinct eigenvalues at which F is defined.

PROOF. To prove the first part of this theorem we shall use the following lemma obtained by Richter [3].

LEMMA 4.1. *Let F be a function which satisfies the hypothesis of (i) of Theorem 4.1, and let X be a matrix for which $F(X)$ is defined. If*

$XB = BX$, then $F(X)B = BF(X)$, that is, $F(X)$ commutes with every matrix which commutes with X .

Any matrix $F(X)$ satisfying the conclusion of the above lemma is a polynomial in X [5].

To prove the second part of the theorem, let F be defined at a matrix Z with distinct eigenvalues. By Theorem 3.1 there exists a $Q = Q(Z)$, H -analytic at Z , such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_i)$, where the λ_i are the eigenvalues of Z ; therefore by (i), $F(Z) = QF(\Lambda)Q^{-1}$ and

$$F(\Lambda) = \sum_{i=1}^n g_i E_{ii}.$$

Therefore, since the g_i are analytic functions of the λ_j and by Corollary 3.1, the λ_j are analytic functions of the z_{rs} , $r, s = 1, \dots, n$, the f_{ij} are analytic functions (in a neighborhood) of the components z_{rs} of Z . Thus F is H -analytic at Z .

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