HAUSDORFF-ANALYTIC FUNCTIONS OF MATRICES[1]

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1. **Introduction.** The concept of Hausdorff-analytic (H-analytic) functions on an (associative) hypercomplex system, over the complex number field, with an identity [1] is, in this paper, applied to functions of matrices with distinct eigenvalues. In the process of showing a sufficient condition for a matric function to be H-analytic in a neighborhood of a matrix Z_0 with distinct eigenvalues, it is shown that the projections (Frobenius covariants) are H-analytic in a neighborhood of Z_0, and that there exists a matric function $Q=Q(Z)$ which is H-analytic in a neighborhood of Z_0 such that $Q^{-1}ZQ=\Lambda=\text{diag }(\lambda_k)$, $\lambda_i\neq\lambda_j$ for $i\neq j$, for all Z in that neighborhood of Z_0.

It has previously been shown [2] that if $f(Z)$ is a function on $\mathfrak{M}$ (the algebra of all square matrices of order n over the complex number field) whose component functions are analytic functions, in some open domain, of the complex (component) variables z_{ij} of $Z=(z_{ij})$, then $f(Z)$ is H-analytic in a corresponding open domain of $\mathfrak{M}$.

2. H-**analyticity of the projections of a matrix with distinct eigenvalues.**

LEMMA 2.1. *The projections (or Frobenius covariants) $P_k(Z)$ corresponding to the eigenvalues λ_k, $k=1,\cdots,s$, of a matrix Z in $\mathfrak{M}$ are given by*

$$P_k(Z) = \frac{1}{2\pi i}\int_{C_k}(\lambda I - Z)^{-1}d\lambda,$$

where C_k is a circle in the complex λ-plane containing λ_k but none of the other λ_j.

PROOF. By [4, p. 22],

$$(\lambda I - Z)^{-1} = \sum_{j=1}^{s} P_j(Z)\sum_{m=0}^{r_j-1}\frac{(Z-\lambda_j I)^m}{(\lambda-\lambda_j)^{m+1}},$$

where r_j is the index of λ_j.

The integral, over a curve in the complex λ-plane, of a matrix of complex functions is given by the matrix of integrals, over the curve, of the matric elements. Therefore if we let C_k be a circle containing

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λ_k but none of the other λ_j, then

$$\frac{1}{2\pi i}\int_{C_k}(\lambda I - Z)^{-1}d\lambda$$
$$= \frac{1}{2\pi i}\sum_{j=1}^{s} P_j(Z)\sum_{m=0}^{r_j-1}(Z - \lambda_j I)^m \int_{C_k}\frac{d\lambda}{(\lambda - \lambda_j)^{m+1}} = P_k(Z).$$

We will use this representation to prove the following.

THEOREM 2.1. *If Z_0 is a matrix with distinct eigenvalues, then there exists a neighborhood N of Z_0 such that for Z in N, $P_k(Z)$ is an H-analytic function of Z.*

PROOF. Let λ_k^0, $k=1, \cdots, n$, be the (distinct) eigenvalues of Z_0, then

$$P_k(Z_0) = \frac{1}{2\pi i}\int_{C_k}(\lambda I - Z)^{-1}d\lambda$$

is the projection of Z_0 corresponding to λ_k^0, where C_k is a sufficiently small circle which has λ_k^0 as its center and all other λ_j^0 in its exterior, that is, if $|\lambda_k^0-\lambda_j^0| > 2\epsilon$ for $j\neq k$, then C_k: $|\lambda-\lambda_k^0| = \epsilon$ will be sufficient.

Now, for all matrices Z sufficiently near Z_0, that is, such that $\text{norm}(Z-Z_0)$ is sufficiently small (where, for convenience, the norm of any matrix $X=(x_{ij})$, $i=1, \cdots, m$, $j=1, \cdots, n$, with complex components, shall be defined by $\text{norm}(X)=\max_{i,j}|x_{ij}|$), the eigenvalues $\lambda_1, \cdots, \lambda_n$ of Z will be near those of Z_0, since the eigenvalues of a matrix are continuous functions of the elements of the matrix [2]. Thus, a neighborhood N of Z_0 may be chosen such that $|\lambda_k-\lambda_k^0| < \epsilon$, $k=1, \cdots, n$, then for each Z in N, Z has distinct eigenvalues λ_j, and λ_k lies within C_k while all other λ_j lie outside C_k. Hence for all Z in N,

$$P_k(Z) = \frac{1}{2\pi i}\int_{C_k}(\lambda I - Z)^{-1}d\lambda.$$

The r, s element of the matrix $P_k(Z)$ is given by

$$P_k(Z)_{rs} = \frac{1}{2\pi i}\int_{C_k}R_{rs}(\lambda, z_{ij})d\lambda,$$

where $R_{rs}(\lambda, z_{ij})$ is the quotient of two polynomials in λ and the z_{ij}, $i, j=1, \cdots, n$. Since C_k does not pass through any of the zeros of $\det(\lambda I-Z)$, regardless of what Z in N is chosen, $R_{rs}(\lambda, z_{ij})$ is a con-continuous function of the complex variables λ and $z_{ij}, i, j=1, \cdots, n$, where each z_{ij} ranges over a region N_{ij} determined by N, and λ lies on C_k; also $R_{rs}(\lambda, z_{ij})$ is an analytic function of each z_{ij} in N_{ij}, for every value of λ of C_k. Therefore, $P_k(Z)_{rs}$ is an analytic function of each z_{ij} of Z in N, and hence, the components $P_k(Z)_{rs}$ are analytic functions of z_{ij}, that is $P_k(Z)$ is H-analytic at Z in N.

3. **The existence of $Q(Z)$.**

LEMMA 3.1. *If Z_0 is a matrix with distinct eigenvalues and x is a vector such that $P_k(Z_0)x\neq 0$ for every $k=1, \cdots, n$, then for Z sufficiently near Z_0, $P_k(Z)x\neq 0$, $k=1, \cdots, n$.*

PROOF. Since $P_k(Z_0)x\neq 0$ for every $k=1, \cdots, n$, there exists a $\delta>0$ such that $\operatorname{norm}(P_k(Z_0)x)>\delta$ for every k. Now $P_k(Z)$ is an H-analytic function of Z in a neighborhood of Z_0 and therefore a continuous function of Z in a neighborhood of Z_0. Thus, for Z near Z_0, $P_k(Z)$ is near $P_k(Z_0)$ and therefore $P_k(Z)x$ is near $P_k(Z_0)x$; in particular Z may be chosen sufficiently close to Z_0 such that

$$\operatorname{norm}(P_k(Z_0)x - P_k(Z)x) < \delta/2,$$

for all k. Hence, for all Z in such a neighborhood N of Z_0,

$$\begin{aligned}\operatorname{norm}(P_k(Z)x) &\geqq \operatorname{norm}(P_k(Z_0)x) - \operatorname{norm}(P_k(Z_0)x - P_k(Z)x)\\ &> \delta - \delta/2 = \delta/2 > 0.\end{aligned}$$

Thus, for Z in N, $P_k(Z)x\neq 0$ for every $k=1, \cdots, n$.

THEOREM 3.1. *Let Z_0 be a matrix with distinct eigenvalues; then there exists a nonsingular matrix Q whose components are analytic functions of the elements of Z, for Z in some neighborhood N of Z_0 (therefore Q and Q^{-1} are H-analytic functions of Z in N), such that $Q^{-1}ZQ=\Lambda=\operatorname{diag}(\lambda_k)$ for all Z in N.*

PROOF. Choose a vector x such that $P_k(Z_0)x\neq 0$ for all $k=1, \cdots, n$, then by Lemma 3.1, for Z sufficiently near Z_0, $Q_k=P_k(Z)x\neq 0$. By [**4**, p. 22], $(Z-\lambda_kI)Q_k=0$, since the index r_k of λ_k is 1 for all k, that is, $ZQ_k=\lambda_kQ_k$. Now, let Q be the matrix whose jth column is Q_j, then $ZQ=Q\Lambda$, where $\Lambda=\operatorname{diag}(\lambda_k)$. Therefore, since the Q_k are linearly independent $Q^{-1}ZQ=\Lambda$.

Since $Q_k=P_k(Z)x$, the components of Q_k are linear combinations of the elements of $P_k(Z)$ and therefore, by Theorem 2.1, they are analytic functions of the elements z_{ij} of Z in a neighborhood of Z_0. Hence Q is H-analytic in a sufficiently small neighborhood N of Z_0.

Also $Q^{-1}=(S_{ij}/\det(Q))$, where S_{ij} is the cofactor of the j, i component of Q. Since $\det(Q)\neq 0$, and S_{ij} and $\det(Q)$ are polynomials with constant coefficients in the analytic components of Q, the components at Q^{-1} are analytic functions of the elements of Z in N and therefore Q^{-1} is also H-analytic in N.

Note: If one knows a Q_0 such that $Q_0^{-1}Z_cQ_0=\Lambda_0=\operatorname{diag}(\lambda_k^0)$ for the given Z_0, then, for the above x, one may choose $x=\sum_{j=1}^n Q_0^{(j)}$, where $Q_0^{(j)}$ is the jth column of Q_0, since $Z_0Q_0^{(k)}=\lambda_k^0Q_0^{(k)}$, that is $P_k(Z_0)Q_0^{(j)}=\delta_{jk}Q_0^{(k)}$, and therefore $P_k(Z_0)x=Q_0^{(k)}\neq 0$ for every $k=1,\cdots,n$.

Corollary 3.1. *If the matrix Z has distinct eigenvalues λ_k, $k=1,\cdots,n$, then the λ_k are analytic functions of the components z_{ij} of Z.*

4. A sufficient condition for H-analyticity of a matric function at a matrix with distinct eigenvalues.

The hypothesis of (i) of the following theorem would be desirable if one wished to view this as a theory for functions of linear transformations of a finite dimensional vector space, for the invariance under similarity transformations, $F(Y)=P^{-1}F(X)P$ for $Y=P^{-1}XP$, permits the definition of a function of a finite dimensional linear transformation to be independent of the choice of basis for the vector space.

Theorem 4.1. *Let $F=\sum_{i,j=1}^n f_{ij}E_{ij}$ be a matric function (where E_{ij} is the $n\times n$ matrix with a 1 in the i, j position and zeros elsewhere).*

(i) *Let F be such that, F defined at X and $Y=P^{-1}XP$ implies that F is defined at Y and $F(Y)=P^{-1}F(X)P$; then, for a given X at which F is defined, $F(X)$ is a polynomial in X. In particular, if $\Lambda=\operatorname{diag}(\lambda_i)$ is a diagonal matrix at which F is defined, then $F(\Lambda)$ is a diagonal matrix, that is,*

$$f_{ij}\Big]_{z_{rs}=\delta_{rs}\lambda_r}=\delta_{ij}g_i(\lambda_1,\cdots,\lambda_n).$$

(ii) *Further, if F is also such that the diagonal component functions f_{ii} are analytic functions of each λ_j at the components of a diagonal matrix with distant eigenvalues λ_i, at which F is defined, that is,*

$$\left.\frac{\partial f_{ii}}{\partial z_{jj}}\right]_{z_{rs}=\delta_{rs}\lambda_r}=\frac{\partial g_i}{\partial\lambda_j}$$

exist for $i,j=1,\cdots,n$, then, F is H-analytic at a matrix Z with distinct eigenvalues at which F is defined.

Proof. To prove the first part of this theorem we shall use the following lemma obtained by Richter [3].

Lemma 4.1. *Let F be a function which satisfies the hypothesis of* (i) *of Theorem* 4.1, *and let X be a matrix for which $F(X)$ is defined. If*

$XB = BX$, then $F(X)B = BF(X)$, that is, $F(X)$ commutes with every matrix which commutes with X.

Any matrix $F(X)$ satisfying the conclusion of the above lemma is a polynomial in X [5].

To prove the second part of the theorem, let F be defined at a matrix Z with distinct eigenvalues. By Theorem 3.1 there exists a $Q = Q(Z)$, H-analytic at Z, such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_i)$, where the λ_i are the eigenvalues of Z; therefore by (i), $F(Z) = QF(\Lambda)Q^{-1}$ and

$$F(\Lambda) = \sum_{i=1}^{n} g_i E_{ii}.$$

Therefore, since the g_i are analytic functions of the λ_j and by Corollary 3.1, the λ_j are analytic functions of the z_{rs}, $r, s = 1, \cdots, n$, the f_{ij} are analytic functions (in a neighborhood) of the components z_{rs} of Z. Thus F if H-analytic at Z.

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