1. Introduction. The concept of Hausdorff-analytic ($H$-analytic) functions on an (associative) hypercomplex system, over the complex number field, with an identity [1] is, in this paper, applied to functions of matrices with distinct eigenvalues. In the process of showing a sufficient condition for a matric function to be $H$-analytic in a neighborhood of a matrix $Z_0$ with distinct eigenvalues, it is shown that the projections (Frobenius covariants) are $H$-analytic in a neighborhood of $Z_0$, and that there exists a matric function $Q = Q(Z)$ which is $H$-analytic in a neighborhood of $Z_0$ such that $Q^{-1}ZQ = \Lambda = \text{diag} (\lambda_k)$, $\lambda_i \neq \lambda_j$ for $i \neq j$, for all $Z$ in that neighborhood of $Z_0$.

It has previously been shown [2] that if $f(Z)$ is a function on $\mathfrak{M}$ (the algebra of all square matrices of order $n$ over the complex number field) whose component functions are analytic functions, in some open domain, of the complex (component) variables $z_{ij}$ of $Z = (z_{ij})$, then $f(Z)$ is $H$-analytic in a corresponding open domain of $\mathfrak{M}$.

2. $H$-analyticity of the projections of a matrix with distinct eigenvalues.

Lemma 2.1. The projections (or Frobenius covariants) $P_k(Z)$ corresponding to the eigenvalues $\lambda_k$, $k = 1, \cdots, s$, of a matrix $Z$ in $\mathfrak{M}$ are given by

$$P_k(Z) = \frac{1}{2\pi i} \int_{C_k} (\lambda I - Z)^{-1} d\lambda,$$

where $C_k$ is a circle in the complex $\lambda$-plane containing $\lambda_k$ but none of the other $\lambda_i$.

Proof. By [4, p. 22],

$$(\lambda I - Z)^{-1} = \sum_{j=1}^{s} P_j(Z) \sum_{m=0}^{r_j-1} \frac{(Z - \lambda_j I)^m}{(\lambda - \lambda_j)^{m+1}},$$

where $r_j$ is the index of $\lambda_j$.

The integral, over a curve in the complex $\lambda$-plane, of a matrix of complex functions is given by the matrix of integrals, over the curve, of the matrix elements. Therefore if we let $C_k$ be a circle containing

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\( \lambda_k \) but none of the other \( \lambda_j \), then

\[
\frac{1}{2\pi i} \int_{c_k} (\lambda I - Z)^{-1} d\lambda = \frac{1}{2\pi i} \sum_{j=1}^{s} \sum_{m=0}^{r_j-1} (Z - \lambda_j I)^m \int_{c_k} \frac{d\lambda}{(\lambda - \lambda_j)^{m+1}} = P_k(Z).
\]

We will use this representation to prove the following.

**Theorem 2.1.** If \( Z_0 \) is a matrix with distinct eigenvalues, then there exists a neighborhood \( N \) of \( Z_0 \) such that for \( Z \) in \( N \), \( P_k(Z) \) is an \( H \)-analytic function of \( Z \).

**Proof.** Let \( \lambda_k, k = 1, \ldots, n \), be the (distinct) eigenvalues of \( Z_0 \), then

\[
P_k(Z_0) = \frac{1}{2\pi i} \int_{c_k} (\lambda I - Z)^{-1} d\lambda
\]

is the projection of \( Z_0 \) corresponding to \( \lambda_0 \), where \( C_k \) is a sufficiently small circle which has \( \lambda_0 \) as its center and all other \( \lambda_0 \) in its exterior, that is, if \( |\lambda_k - \lambda_0| > 2e \) for \( j \neq k \), then \( C_k: |\lambda - \lambda_0| = e \) will be sufficient.

Now, for all matrices \( Z \) sufficiently near \( Z_0 \), that is, such that norm(\( Z - Z_0 \)) is sufficiently small (where, for convenience, the norm of any matrix \( X = (x_{ij}) \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \), with complex components, shall be defined by norm\( (X) = \max_{i,j} |x_{ij}| \)), the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( Z \) will be near those of \( Z_0 \), since the eigenvalues of a matrix are continuous functions of the elements of the matrix \( [2] \). Thus, a neighborhood \( N \) of \( Z_0 \) may be chosen such that \( |\lambda_k - \lambda_0| < e, k = 1, \ldots, n \), then for each \( Z \) in \( N \), \( Z \) has distinct eigenvalues \( \lambda_j \), and \( \lambda_k \) lies within \( C_k \) while all other \( \lambda_j \) lie outside \( C_k \). Hence for all \( Z \) in \( N \),

\[
P_k(Z) = \frac{1}{2\pi i} \int_{c_k} (\lambda I - Z)^{-1} d\lambda.
\]

The \( r, s \) element of the matrix \( P_k(Z) \) is given by

\[
P_k(Z)_{rs} = \frac{1}{2\pi i} \int_{c_k} R_{rs}(\lambda, z_{ij}) d\lambda,
\]

where \( R_{rs}(\lambda, z_{ij}) \) is the quotient of two polynomials in \( \lambda \) and the \( z_{ij}, i, j = 1, \ldots, n \). Since \( C_k \) does not pass through any of the zeros of \( \det(\lambda I - Z) \), regardless of what \( Z \) in \( N \) is chosen, \( R_{rs}(\lambda, z_{ij}) \) is a continuous function of the complex variables \( \lambda \) and \( z_{ij}, i, j = 1, \ldots, n \),
where each $z_{ij}$ ranges over a region $N_{ij}$ determined by $N$, and $\lambda$ lies on $C_k$; also $R_{\alpha}(\lambda, z_{ij})$ is an analytic function of each $z_{ij}$ in $N_{ij}$, for every value of $\lambda$ of $C_k$. Therefore, $P_k(Z)_{\alpha}$ is an analytic function of each $z_{ij}$ of $Z$ in $N$, and hence, the components $P_k(Z)_{\alpha}$ are analytic functions of $z_{ij}$, that is $P_k(Z)$ is $H$-analytic at $Z$ in $N$.

3. The existence of $Q(Z)$.

**Lemma 3.1.** If $Z_0$ is a matrix with distinct eigenvalues and $x$ is a vector such that $P_k(Z_0)x \neq 0$ for every $k = 1, \cdots, n$, then for $Z$ sufficiently near $Z_0$, $P_k(Z)x \neq 0$, $k = 1, \cdots, n$.

**Proof.** Since $P_k(Z_0)x \neq 0$ for every $k = 1, \cdots, n$, there exists a $\delta > 0$ such that $\text{norm}(P_k(Z_0)x) > \delta$ for every $k$. Now $P_k(Z)$ is an $H$-analytic function of $Z$ in a neighborhood of $Z_0$ and therefore a continuous function of $Z$ in a neighborhood of $Z_0$. Thus, for $Z$ near $Z_0$, $P_k(Z)$ is near $P_k(Z_0)$ and therefore $P_k(Z)x$ is near $P_k(Z_0)x$; in particular $Z$ may be chosen sufficiently close to $Z_0$ such that

$$\text{norm}(P_k(Z)x - P_k(Z_0)x) < \delta/2,$$

for all $k$. Hence, for all $Z$ in such a neighborhood $N$ of $Z_0$,

$$\text{norm}(P_k(Z)x) \geq \text{norm}(P_k(Z_0)x) - \text{norm}(P_k(Z_0)x - P_k(Z)x)$$

$$> \delta - \delta/2 = \delta/2 > 0.$$  

Thus, for $Z$ in $N$, $P_k(Z)x \neq 0$ for every $k = 1, \cdots, n$.

**Theorem 3.1.** Let $Z_0$ be a matrix with distinct eigenvalues; then there exists a nonsingular matrix $Q$ whose components are analytic functions of the elements of $Z$, for $Z$ in some neighborhood $N$ of $Z_0$ (therefore $Q$ and $Q^{-1}$ are $H$-analytic functions of $Z$ in $N$), such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_k)$ for all $Z$ in $N$.

**Proof.** Choose a vector $x$ such that $P_k(Z_0)x \neq 0$ for all $k = 1, \cdots, n$, then by Lemma 3.1, for $Z$ sufficiently near $Z_0$, $Q_k = P_k(Z)x \neq 0$. By [4, p. 22], $(Z - \lambda_k I)Q_k = 0$, since the index $r_k$ of $\lambda_k$ is 1 for all $k$, that is, $ZQ_k = \lambda_k Q_k$. Now, let $Q$ be the matrix whose $j$th column is $Q_j$, then $ZQ = QA$, where $A = \text{diag}(\lambda_k)$. Therefore, since the $Q_k$ are linearly independent $Q^{-1}ZQ = \Lambda$.

Since $Q_k = P_k(Z)x$, the components of $Q_k$ are linear combinations of the elements of $P_k(Z)$ and therefore, by Theorem 2.1, they are analytic functions of the elements $z_{ij}$ of $Z$ in a neighborhood of $Z_0$. Hence $Q$ is $H$-analytic in a sufficiently small neighborhood $N$ of $Z_0$.

Also $Q^{-1} = (S_{ij}/\text{det}(Q))$, where $S_{ij}$ is the cofactor of the $j$, $i$ component of $Q$. Since $\text{det}(Q) \neq 0$, and $S_{ij}$ and $\text{det}(Q)$ are polynomials.
with constant coefficients in the analytic components of $Q$, the components at $Q^{-1}$ are analytic functions of the elements of $Z$ in $N$ and therefore $Q^{-1}$ is also $H$-analytic in $N$.

Note: If one knows a $Q_0$ such that $Q_0^{-1}Z_0Q_0 = \Lambda_0 = \text{diag}(\lambda_i^0)$ for the given $Z_0$, then, for the above $x$, one may choose $x = \sum_{j=1}^{n} Q_0^{(j)}$, where $Q_0^{(j)}$ is the $j$th column of $Q_0$, since $Z_0Q_0^{(k)} = \lambda_i^0Q_0^{(k)}$, that is $P_k(Z_0)Q_0^{(j)} = \delta_{jk}Q_0^{(k)}$, and therefore $P_k(Z_0)x = Q_0^{(k)} \neq 0$ for every $k = 1, \ldots, n$.

**Corollary 3.1.** If the matrix $Z$ has distinct eigenvalues $\lambda_k$, $k = 1, \ldots, n$, then the $\lambda_k$ are analytic functions of the components $z_{ij}$ of $Z$.

4. **A sufficient condition for $H$-analyticity of a matrix function at a matrix with distinct eigenvalues.** The hypothesis of (i) of the following theorem would be desirable if one wished to view this as a theory for functions of linear transformations of a finite dimensional vector space, for the invariance under similarity transformations, $F(Y) = P^{-1}F(X)P$ for $Y = P^{-1}XP$, permits the definition of a function of a finite dimensional linear transformation to be independent of the choice of basis for the vector space.

**Theorem 4.1.** Let $F = \sum_{i,j=1}^{n} f_{ij}E_{ij}$ be a matrix function (where $E_{ij}$ is the $n \times n$ matrix with a 1 in the $i, j$ position and zeros elsewhere).

(i) Let $F$ be such that, $F$ defined at $X$ and $Y = P^{-1}XP$ implies that $F$ is defined at $Y$ and $F(Y) = P^{-1}F(X)P$; then, for a given $X$ at which $F$ is defined, $F(X)$ is a polynomial in $X$. In particular, if $\Lambda = \text{diag}(\lambda_i)$ is a diagonal matrix at which $F$ is defined, then $F(\Lambda)$ is a diagonal matrix, that is,

$$f_{ij}]_{x_i = \delta_{ir} \lambda_r} = \delta_{ij}g_i(\lambda_1, \ldots, \lambda_n).$$

(ii) Further, if $F$ is also such that the diagonal component functions $f_{ii}$ are analytic functions of each $\lambda_i$ at the components of a diagonal matrix with distant eigenvalues $\lambda_i$, at which $F$ is defined, that is,

$$\left. \frac{\partial f_{ii}}{\partial z_{jj}} \right|_{x_i = \delta_{ir} \lambda_r} = \frac{\partial g_i}{\partial \lambda_j},$$

exist for $i, j = 1, \ldots, n$, then, $F$ is $H$-analytic at a matrix $Z$ with distinct eigenvalues at which $F$ is defined.

**Proof.** To prove the first part of this theorem we shall use the following lemma obtained by Richter [3].

**Lemma 4.1.** Let $F$ be a function which satisfies the hypothesis of (i) of Theorem 4.1, and let $X$ be a matrix for which $F(X)$ is defined. If
$XB = BX$, then $F(X)B = BF(X)$, that is, $F(X)$ commutes with every matrix which commutes with $X$.

Any matrix $F(X)$ satisfying the conclusion of the above lemma is a polynomial in $X$ [5].

To prove the second part of the theorem, let $F$ be defined at a matrix $Z$ with distinct eigenvalues. By Theorem 3.1 there exists a $Q = Q(Z)$, $H$-analytic at $Z$, such that $Q^{-1}ZQ = \Lambda = \text{diag}(\lambda_i)$, where the $\lambda_i$ are the eigenvalues of $Z$; therefore by (i), $F(Z) = QF(\Lambda)Q^{-1}$ and

$$F(\Lambda) = \sum_{i=1}^{n} g_i E_{ii}.$$ 

Therefore, since the $g_i$ are analytic functions of the $\lambda_i$ and by Corollary 3.1, the $\lambda_i$ are analytic functions of the $z_{rs}$, $r$, $s = 1, \ldots, n$, the $f_{ij}$ are analytic functions (in a neighborhood) of the components $z_{rs}$ of $Z$. Thus $F$ if $H$-analytic at $Z$.

References


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