FUNCTIONAL EQUATIONS INVOLVING A PARAMETER[1]

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1. The present note concerns the examination of nonlinear functional equations depending on a parameter. We investigate here the iterative method described in paper [1] and [2], which is a generalization of Newton's classical method. Another abstract formalism for Newton's method has been given first by L. V. Kantorovich (for references see [4]) and applied by him to the examination of operator equations in Banach spaces.

The main point here is the application of the majorant method, which was used by Kantorovich [4] and also in paper [3].

The results stated here make it possible to find an error estimation of the exact solution in the case when the solution of a suitable approximate equation is given.

An application to approximate solutions of operator equations in Hilbert space will be given in another note.

Let X and M be two Banach spaces, and let $F(x, \mu)$ be a nonlinear continuous functional defined on the space $X \dotplus M$, where x and μ are in some closed spheres in X, M with centres x_0, μ_0, respectively.

Consider the nonlinear functional equation

$$F(x, \mu) = 0. \tag{1}$$

Let us assume that $F(x, \mu)$ is differentiable in Fréchet's sense in the spheres mentioned above with respect to each of the two variables x, μ separately. Denote by

$$f(x, \mu) = F'(x, \mu) = F'_x(x, \mu)$$

the partial Fréchet derivative of $F(x, \mu)$.

Putting

$$f_n = f(x_n, \mu) = F'_x(x_n, \mu)$$

we choose a sequence of elements $y_n \in X$, $\mu \in M$ such that

$$\|y_n\| = 1, \qquad f_n(y_n, \mu) = \|f_n\|, \qquad n = 0, 1, 2, \cdots \tag{2}$$

provided that such a choice is possible.

The iterative process for solving equation (1) is defined as in papers [1] and [2]:

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(3) $$x_1(\mu) = x_0 - \frac{F(x_0, \mu)}{f_0(y_0, \mu)} y_0;$$

$$x_{n+1}(\mu) = x_n(\mu) - \frac{F(x_n, \mu)}{f_n(y_n, \mu)} y_n.$$

Let us further assume that the second Fréchet derivative $F''(x, \mu) = F''_{xx}(x, \mu)$ of $F(x, \mu)$ exists for x in some sphere of X with centre x_0 and that the derivatives $\partial F(x, \mu)/\partial\mu$, $\partial F'(x_0, \mu)/\partial\mu$ and $\partial F''(x, \mu)/\partial\mu$ exist where μ belongs to some sphere of M with centre μ_0.

Consider now the real equation

(4) $$Q(z, \nu) = 0,$$

where $Q(z, \nu)$ is a real function of the real variables z, ν, being twice continuously differentiable in the intervals (z_0, z') and (ν_0, ν'). Put $Q'(z, \nu) = Q_z'(z, \nu)$ and $Q''(z, \nu) = Q''_{z^2}(z, \nu)$.

Following the argument of paper [3] let us say that equation (1) possesses a real majorant equation (4), if the following conditions are satisfied:

(1°) $$Q'(z_0, \nu) \neq 0 \quad \text{and} \quad B = -\frac{1}{Q'(z_0, \nu)} > 0;$$

(2°) $$\|F(x_0, \mu)\| \leqq Q(z_0, \nu);$$

(3°) $$\frac{1}{\|F'(x_0, \mu)\|} \leqq B;$$

(4°) $\|F''(x, \mu)\| \leqq Q''(z, \nu)$ if $\|x - x_0\| \leqq z - z_0 \leqq z' - z_0$, provided that μ and ν are fixed.

The following theorem of paper [3] will be used in the sequel:

THEOREM (a). *If for fixed μ and ν equation* (1) *possesses a real majorant equation* (4), *and if equation* (4) *has a real root z^* in the segment* (z_0, z'), *then equation* (1) *has a solution x^*, where $\|x^* - x_0\| \leqq z' - z_0$, and the sequence of approximate solutions x_n constructed by process* (3) *converges to it. Moreover, we have the estimate*

(5) $$\|x_n - x^*\| \leqq z^* - z_n,$$

where z_n is defined by Newton's classical process, i.e.

(6) $$z_{n+1}(\nu) = z_n(\nu) - \frac{Q(z_n, \nu)}{Q'(z_n, \nu)}.$$

Suppose now that the approximate solution x_0 of equation (1) is given for a certain value μ_0 of the parameter and we are interested in the solution of this equation for some other value μ of the parameter. The following theorem concerns this case.

THEOREM 1. *Let us assume that the following conditions are satisfied*:

$$(1°) \qquad Q'_z(z_0, \nu_0) \neq 0 \quad \textit{and} \quad B = -\frac{1}{Q'(z_0, \nu_0)} > 0;$$

$$(2°) \qquad |F(x_0, \mu_0)| \leqq Q(z_0, \nu_0);$$

$$(3°) \qquad \frac{1}{\|F'(x_0, \mu_0)\|} \leqq B;$$

$$(4°) \qquad \|F''(x, \mu_0)\| \leqq Q''(z, \nu_0) \qquad \textit{if } \|x - x_0\| \leqq z - z_0 \leqq z' - z_0;$$

$$(5°) \qquad \left\|\frac{\partial}{\partial\mu} F(x_0, \mu)\right\| \leqq \frac{\partial}{\partial\nu} Q(z_0, \nu) \qquad \textit{if } \|\mu - \mu_0\| \leqq \nu - \nu_0 \leqq \nu' - \nu_0;$$

$$(6°) \qquad \left\|\frac{\partial}{\partial\mu} F'(x_0, \mu)\right\| \leqq \frac{\partial}{\partial\nu} Q'(z_0, \nu) \qquad \textit{if } \|\mu - \mu_0\| \leqq \nu - \nu_0 \leqq \nu' - \nu_0;$$

$$(7°) \qquad \left\|\frac{\partial}{\partial\mu} F''(x, \mu)\right\| \leqq \frac{\partial}{\partial\nu} Q''(z, \nu) \qquad \textit{if } \|\mu - \mu_0\| \leqq \nu - \nu_0 \leqq \nu' - \nu_0$$

$$\textit{and } \|x - x_0\| \leqq z - z_0 \leqq z' - z_0.$$

If equation (4) *possesses a real solution* $z(\nu)$, $(z_0 \leqq z(\nu) \leqq z')$, *for some* ν, $(\nu_0 \leqq \nu \leqq \nu')$, *then equation* (1) *has a solution* $x(\mu)$ *if* $\|\mu - \mu_0\| \leqq \nu - \nu_0 \leqq \nu' - \nu_0$ *and the sequence of approximate solutions* $x_n(\mu)$ *defined by process* (3) *converges to it. Moreover, we have*

$$\|x(\mu) - x_0\| \leqq z(\nu) - z_0.$$

PROOF. In order to prove this theorem it is sufficient to show that the conditions of Theorem (a) are satisfied. First of all we shall show that condition (2°) of the preceding theorem is fulfilled. In fact, we have by (5°)

$$\begin{aligned} |F(x_0, \mu)| &= \left|F(x_0, \mu_0) + \int_{\mu_0}^{\mu} \frac{\partial}{\partial\mu} F(x_0, \bar\mu) d\bar\mu\right| \\ &\leqq Q(z_0, \nu_0) + \int_{\nu_0}^{\nu} \frac{\partial}{\partial\nu} Q(z_0, \bar\nu) d\bar\nu \\ &= Q(z_0, \nu_0) + Q(z_0, z) - Q(z_0, \nu_0) = Q(z_0, \nu). \end{aligned}$$

Further, we get by (6°), (1°) and (3°)

$$\begin{aligned}\|F'(x_0, \mu)\| &= \left\|F'(x_0, \mu_0) + \int_{\mu_0}^{\mu} \frac{\partial}{\partial \bar{\mu}} F'(x_0, \bar{\mu}) d\bar{\mu}\right\| \\ &\geqq \|F'(x_0, \mu_0)\| - \left\|\int_{\mu_0}^{\mu} \frac{\partial}{\partial \bar{\mu}} F'(x_0, \bar{\mu}) d\bar{\mu}\right\| \\ &\geqq \|F'(x_0, \mu_0)\| \left(1 - \left\|\int_{\mu_0}^{\mu} \frac{\partial}{\partial \bar{\mu}} F'(x_0, \bar{\mu}) d\bar{\mu}\right\| \frac{1}{\|F'(x_0, \mu_0)\|}\right) \\ &\geqq \|F'(x_0, \mu_0)\| \left(1 - \frac{\int_{\nu_0}^{\nu} \frac{\partial}{\partial \bar{\nu}} Q'(x_0, \bar{\nu}) d\bar{\nu}}{\|F'(x_0, \mu_0)\|}\right) \\ &\geqq \|F'(x_0, \mu_0)\| \left(1 + \frac{Q'(z_0, \nu) - Q'(z_0, \nu_0)}{Q'(z_0, \nu_0)}\right) \\ &= \frac{\|F'(x_0, \mu_0)\|}{Q'(z_0, \nu_0)} Q'(z_0, \nu) \geqq - Q'(z_0, \nu),\end{aligned}$$

if the last expression is positive.

We have now to prove that $Q'(z_0, \nu)$ is negative. For this purpose we shall show that $Q''(z, \nu)$ is non-negative. We have by (7°)

$$\begin{aligned}\| F''(x, \mu)\| &\leqq \|F''(x, \mu_0)\| + \left\|\int_{\mu_0}^{\mu} \frac{\partial}{\partial \bar{\mu}} F''(x, \bar{\mu}) d\bar{\mu}\right\| \\ &\leqq Q''(z, \nu_0) + \int_{\nu_0}^{\nu} \frac{\partial}{\partial \bar{\nu}} Q''(x, \bar{\nu}) d\bar{\nu} \\ &= Q''(z, \nu_0) + Q''(z, \nu) - Q''(z, \nu_0) \\ &= Q''(z, \nu).\end{aligned}$$

If $Q'(z_0, \nu)$ were non-negative we should have $Q'(z, \nu) \geqq 0$ since $Q''(z, \nu) \geqq 0$. Hence we get by (1°) and (5°) $Q(z, \nu) \geqq Q(z_0, \nu) \geqq Q(z_0, \nu_0) > 0$. But this leads to a contradiction because equation (4) has a real solution. Thus, we conclude that condition (1°) is satisfied. It remains to prove that condition (4°) of Theorem (a) is also satisfied, i.e. $\|F''(x, \mu)\| \leqq Q''(z, \nu)$ if $\|x - x_0\| \leqq z - z_0 \leqq z' - z_0$, and $\|\mu - \mu_0\| \leqq \nu - \nu_0 \leqq \nu' - \nu_0$. But this verification has already been obtained above, and thus the theorem is proved.

Remark 1. The error estimate is given by the formula

$$\|x_n(\mu) - x(\mu)\| \leqq z(\nu) - z_n(\nu).$$

This remark follows from (5).

Remark 2. Condition (2°) can be replaced by condition

$$\| x_1(\mu) - x_0\| \leqq z_1(\nu) - z_0.$$

This remarks follows from the proof of Theorem (a).

Consider now the following particular case of a functional equation depending on a parameter:

$$F(x, \mu) = G(x) + \mu H(x) = 0, \tag{7}$$

where $G(x)$ and $H(x)$ are nonlinear, continuous functionals on X and μ is a real number. Suppose that a solution of equation (7) is given for $\mu_0 = 0$. Applying Theorem 1 we obtain the following

THEOREM 2. *Let us assume that $G(x)$ and $H(x)$ are twice continuously differentiable in the sense of Fréchet and the following conditions are fulfilled*:

(1) $$G(x_0) = 0.$$

(2) $$\frac{1}{\|G'(x_0)\|} \leqq B.$$

(3) $$\|G''(x)\| \leqq K \qquad \text{if } \|x - x_0\| \leqq z' - z_0;$$

(4) $$| H(x_0) | \leqq \eta;$$

(5) $$\|H'(x_0)\| \leqq \alpha;$$

(6) $$\|H''(x)\| \leqq \beta \qquad \text{if } \|x - x_0\| \leqq z' - z_0,$$

and

(7) $$\frac{(1 - \alpha B\nu)^2}{B^2} - 2\nu\eta(K + \nu\beta) \geqq 0; \qquad 0 < \alpha B\nu < 1.$$

Then equation (7) *has a solution if* $|\mu| \leqq \nu$ *and the sequence of approximate solutions* x_n *defined by process* (3) *converges to it. Moreover, the solution* x^* *satisfies the inequality* $\|x^* - x_0\| \leqq z(\nu)$ *and conditions* (5) *and* (6) *hold, provided that the majorant equation* (4) *is replaced by the following one*:[2]

(8) $$Q(z, \nu) = \frac{K + \nu\beta}{2} z^2 - \frac{1 - \alpha B\nu}{B} z + \nu\eta = 0, \qquad (z_0 = 0, \nu_0 = 0).$$

PROOF. It is easy to verify that all conditions (1°)–(7°) of Theorem 1 are satisfied.

REMARK 3. Instead of the majorant equation (8) we can use the following one[2]

[2] It seems to be interesting to notice that these majorant equations are the same as those considered by Kantorovich [4].

(9) $$Q(z, \nu) = \frac{K}{2} z^2 + \nu \int_0^z dz_1 \int_0^{z_1} \beta(t)dt - \frac{1 - \alpha B\nu}{B} z + \nu\eta = 0.$$

In this case condition (6) should be replaced by condition (10)

(10) $$\| H''(x)\| \leqq \beta(r) \text{ if } \|x - x_0\| \leqq r.$$

All assertions of Theorem 2 hold if equation (9) has a positive root for $\|\mu\| \leqq \nu$.

REMARK 4. Notice that Corollary 2 in [3, p. 23] may be considered as a particular case of Theorem 2 if we put

$$F(x, \mu) = [F(x) - F(x_0)] + \mu F(x_0), \qquad (\mu_0 = 1).$$

2. In this section we are concerned with the error estimation for the approximate solution of the functional equation

(11) $$F(x) = 0,$$

where $F(x)$ is a nonlinear continuous functional defined on the Banach space X.

At the same time we consider the approximate functional equation

(12) $$G(x) = 0,$$

where $G(x)$ is also a nonlinear continuous functional defined on X. Suppose that x_0 is a solution of equation (12). In order to find how near the solution of equation (11) is to x_0 we introduce the following functional equation depending on a parameter:

(13) $$F(x, \mu) = G(x) + \mu[F(x) - G(x)] = G(x) + \mu H(x) = 0.$$

Suppose that both $F(x)$ and $G(x)$ are twice continuously differentiable in the sense of Fréchet. We are now in a position to apply Theorem 2. Hence we get

COROLLARY 1. *Let us assume that the following conditions are fulfilled.*

(1) $G(x_0) = 0$,
(2) $1/\|G'(x_0)\| \leqq B$,
(3) $\|G''(x)\| \leqq K$ *if* $\|x - x_0\| \leqq z' - z_0$,
(4) $|F(x_0)| \leqq \eta$,
(5) $\|F'(x_0) - G'(x_0)\| \leqq \alpha$,
(6) $\|F''(x) - G''(x)\| \leqq \beta$ *if* $\|x - x_0\| \leqq z' - z_0$,
(7) $(1 - \alpha B)^2/B^2 - 2\eta(K + \beta) \geqq 0$, $(\alpha B \leqq 1)$.

Then equation (11) *has a solution* x^* *such that*

$$\| x^* - x_0\| \leqq z_1,$$

where z_1 is the smallest root of the equation

$$\frac{K+\beta}{2}z^2 - \frac{1-\alpha B}{B}z + \eta = 0.$$

This estimation may be useful especially in the case, when the expression (2) is more simple than the corresponding one for the functional F. We shall now apply the estimation obtained above replacing $G(x)$ by

(14) $$G(x) = F(x_0) + F'(x_0)(x - x_0).$$

As the initial approach, which appears in Corollary 1, we take now the solution x_1 of equation

(15) $$G(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) = 0.$$

Condition (15), is, of course, satisfied if x_1 is defined by process (3). As a particular case of Corollary 1 we obtain

COROLLARY 2. *Let us assume that the following conditions are satisfied:*

(1) $|F(x_0)| \leqq \eta$,
(2) $1/\|F'(x_0)\| \leqq B$,
(3) $\|F''(x)\| \leqq K$ *if* $\|x - x_0\| \leqq z' - z_0$,
(4) $|F(x_1)| \leqq \eta_1$,
(5) $(1 - KB^2\eta_1)^2/B^2 - 2\eta_1 K \geqq 0$, $(KB^2\eta_1 \leqq 1)$.

Then equation $F(x) = 0$ has a solution x^* such that

$$\|x^* - x_1\| \leqq z_1,$$

where z_1 is the smallest root of equation

$$\frac{1}{2}Kz^2 - \frac{1 - B^2K\eta}{B}z - \eta_1 = 0.$$

Let us observe that in this case the following conditions are satisfied:

(1) $G(x_1) = 0$,
(2) $1/\|G'(x_1)\| = 1/\|F'(x_0)\| \leqq B$,
(3) $\|G''(x)\| = 0$,
(4) $|F(x_1)| \leqq \eta_1$,
(5) $\|F'(x_1) - G'(x_1)\| = \|F'(x_1) - F'(x_0)\| \leqq K\|x_1 - x_0\| \leqq KB\eta = \alpha$,
(6) $\|F''(x) - G''(x)\| = \|F''(x)\| \leqq K = \beta$.

But this means that all conditions (1)–(7) of Corollary 1 are satisfied provided that x_0 is replaced by x_1 and $\alpha = KB\eta$, $\beta = K$.

References

1. M. Altman, *On the approximate solution of non-linear functional equations*, Bull. Acad. Polon. Sci. Cl. III vol. 5 (1957) pp. 457–460.

2. ———, *Concerning approximate solutions of non-linear functional equations*, Bull. Acad. Polon. Sci. Cl. III vol. 5 (1957) pp. 461–465.

3. ———, *On the approximate solutions of non-linear functional equations in Banach spaces*, Bull. Acad. Polon. Sci. Ser. Math. vol. 6 (1958) pp. 19–24.

4. L. V. Kantorovich, *Some further applications of Newton's method to functional equations* (in Russian), Vestnik Leningrad. Univ. Ser. Mat. vol. 2 (1957) pp. 68–103.

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AN UNCOUNTABLE SET OF INCOMPARABLE DEGREES

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The purpose of this note is to prove the following:[1]

THEOREM. *There is an uncountable set of pairwise incomparable degrees of recursive unsolvability.*

By Zorn's lemma, there is a maximal set of pairwise incomparable degrees of recursive unsolvability different from $\mathbf{0}$; we must show that this set is not countable. Hence our theorem follows from:

LEMMA. *If* $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots$ *is a sequence of degrees different from* $\mathbf{0}$, *then there is a degree* $\boldsymbol{b}$ *which is incomparable with each* $\boldsymbol{a}_n$.

PROOF.[2] Let α_n be a function of degree $\boldsymbol{a}_n$; we shall construct a function β of degree $\boldsymbol{b}$. As in [1], β is constructed by defining inductively a function κ such that $\kappa(a) = \bar{\beta}(\nu(a))$ with $\nu(a) = lh(\kappa(a))$; κ and ν must satisfy the conditions that $\kappa(a)$ is a sequence number, $\kappa(a+1)$ extends $\kappa(a)$, and $\nu(a+1) > \nu(a)$. We then have $\beta(a) = (\kappa(a+1))_a - 1$.

Let $\kappa(0) = 1$. To define $\kappa(a+1)$, let $n = (a)_1$ and $e = (a)_2$. If a is even, set

$$\kappa(a+1) = \kappa(a)\cdot p_{\nu(a)} \exp(\{e\}^{\alpha_n}(\nu(a)) + 2)$$

if $\{e\}^{\alpha_n}(\nu(a))$ is defined, and $\kappa(a+1) = \kappa(a)\cdot p_{\nu(a)}$ otherwise. Then clearly $\beta \neq \{e\}^{\alpha_n}$ for any function β such that $\beta(\nu(a+1)) = \kappa(a+1)$.

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[2] We use the notation of [1] in the proof.