SOME GLOBAL PROPERTIES OF HYPERSURFACES

ROBERT E. STONG

1. Introduction. The translation theorem of Hopf [1] has been extended by Hsiung [2] and Voss [4] independently to hypersurfaces and by Hsü [3] to other elementary transformations. The purpose of this paper is to extend to hypersurfaces in \((n+1)\)-dimensional Euclidean space some results obtained by Hsü [3] for the case \(n=2\).

All hypersurfaces mentioned will be assumed to be twice differentiably imbedded in an \((n+1)\)-dimensional Euclidean space \(E^{n+1}(n+1 \geq 3)\). The notation used will be that of Hsiung [2]. In particular, \(X, N, M_1, A\) denote the position vector, unit inner normal, first mean curvature, and area for the hypersurface \(V^n\). Corresponding quantities for other hypersurfaces will be denoted by *, or by primes.

Considerable use will be made of the vector product defined by Hsiung [2]. Namely, if \(i_1, \ldots, i_{n+1}\) denotes a fixed frame of mutually orthogonal unit vectors and \(A_1, \ldots, A_n\) are \(n\) vectors whose components in this frame are \(A_{i}^\alpha (i=1, \ldots, n; \alpha=1, \ldots, n+1)\), the vector product is defined by

\[
A_1 \times \cdots \times A_n = (-1)^n \begin{vmatrix}
i_1 & i_2 & \cdots & i_{n+1} \\
A_1^1 & A_2^1 & \cdots & A_{n+1}^1 \\
\cdots & \cdots & \cdots & \cdots \\
A_1^n & A_2^n & \cdots & A_{n+1}^n 
\end{vmatrix}.
\]

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Further, in any vector product involving differentials, the exterior convention for multiplication of differentials will be observed. As in [2], the following formulae will be used:

\[(1.1) \quad dX \times \cdots \times dX = n!N dA,\]
\[(1.2) \quad dX \times \cdots \times dX \times dN = - n!M_1 N dA.\]

A closed nonselfintersecting hypersurface \( V^n \) is said to be convex with respect to the point \( O \) if: (1) every straight line through \( O \) meets \( V^n \) in at most two points, and (2) the mapping \( f: V^n \to V^n \) which takes a point of \( V^n \) into the other point on the same line through \( O \) is a differentiable homeomorphism.

The following results will be obtained:

**Theorem 1.** Let \( V^n, V^{*n} \) be two closed orientable hypersurfaces. Suppose there is a differentiable homeomorphism \( f: V^n \to V^{*n} \) such that:

1. each straight line \( PP^* \) joining corresponding points \( P \) and \( P^* \) passes through a fixed point \( O \);
2. with \( O \) as origin, the position vectors and first mean curvatures are related to each other by either (a) \( M_1^* X^* = M_1 X \) or by (b) \( M_1^* X^* = - M_1 X \) throughout \( V^n \) and \( V^{*n} \); and
3. \( V^n \) and \( V^{*n} \) contain no pieces of hypercones with vertex \( O \). Then \( f \) is a homothetic transformation with center \( O \) and positive or negative constant of proportionality as (a) or (b) holds.

**Theorem 2.** Let \( V^n \) and \( V^{*n} \) be two closed orientable hypersurfaces. Suppose there is a differentiable homeomorphism \( f: V^n \to V^{*n} \) such that:

1. there is a fixed point \( O \) such that for every pair of points \( P, Q \) of \( V^n \) and their images \( P^*, Q^* \) in \( V^{*n} \), the angles \( POQ \) and \( P^*OQ^* \) are equal;
2. with \( O \) as origin \( M_1^* X^* \) and \( M_1 X \) are equal in magnitude; and
3. neither \( V^n \) nor \( V^{*n} \) contains pieces of hypercones with vertex \( O \). Then \( f \) is a similarity transformation with \( O \) as center of similitude.

**Theorem 3.** Let \( V^n, V^{*n} \) be two closed orientable hypersurfaces. Suppose there is a differentiable homeomorphism \( f: V^n \to V^{*n} \) such that:

1. each straight line \( PP^* \) joining corresponding points \( P \) and \( P^* \) passes through a fixed point \( O \);
2. with \( O \) as origin, the position vectors, normals, and first mean curvatures are related by either

\[(a) \quad M_1^* X^* = - \left( M_1 + 2 \frac{X \cdot N}{X \cdot X} \right) X\]

or by

\[(b) \quad M_1^* X^* = \left( M_1 + 2 \frac{X \cdot N}{X \cdot X} \right) X\]
throughout $V^n$ and $V^*_n$; and (3) neither $V^n$ nor $V^*_n$ contains the point $O$ or pieces of hypercones with vertex $O$. Then $f$ is an inversion with center $O$ and real or imaginary radius of inversion as (a) or (b) respectively holds.

2. Integral formulae. Let $V^n$ be an orientable hypersurface with closed boundary $V^{n-1}$. Further suppose $V^n$ does not contain $O$. Let $k$ be a differentiable function on $V^n$ which is finite and nonzero. Let a hypersurface $V^*_n$ be defined by

(2.1) \[ X^* = kX, \]

with

(2.2) \[ kM^*_1 = M_1. \]

Applying (1.1) and noting that any vector product having two or more factors of $X$ vanishes, one finds

(2.3) \[ n!N^*dA^* = n!k^*NdA + nk^{n-1}(Xdk \times dX \times \cdots \times dX). \]

Taking the scalar product of (2.3) with $M^*_1X^* = M_1X$ and using $X \cdot (Xdk \times dX \times \cdots \times dX) = 0$ yield

\[ M^*_1X^* \cdot N^*dA^* = k^nM_1X \cdot NdA. \]

Let $\alpha = N \cdot (X \times dX \times \cdots \times dX)$ and $\beta = N^* \cdot (X \times dX \times \cdots \times dX)$. Then it follows that

(2.4) \[ d\alpha = n!M_1X \cdot NdA + n!dA. \]

From (1.2) one has

\[ -n!M^*_1X^* \cdot N^*dA^* = X^* \cdot (dN^* \times dX^* \times \cdots \times dX*); \]

so that

(2.5) \[ \beta = n!k^{-n}M^*_1X^* \cdot N^*dA^* + n!N^* \cdot NdA \]

\[ = n!M_1X \cdot NdA + n!N^* \cdot NdA. \]

Subtracting (2.5) from (2.4) gives

(2.6) \[ n!(1 - N^* \cdot N)dA = d(\alpha - \beta). \]

Integrating (2.6) and applying Stokes' formula yields

(2.7) \[ n! \int_{V^n} (1 - N^* \cdot N)dA \]

\[ = \int_{V^{n-1}} (N - N^*) \cdot (X \times dX \times \cdots \times dX). \]
If (2.1) is replaced by

\[ X^* = -kX, \]

with (2.2) unchanged, one finds as above that

\[ n! \int_{V^n} (1 + N^* \cdot N) dA \]

\[ = \int_{V^{n-1}} (N + N^*) \cdot (X \times dX \times \cdots \times dX). \]

3. Proofs of theorems.

Proof of Theorem 1. Let \( M_i X^* = M_i X \) and let \( X^* = kX \) where \( k = M_i / M_i^* \).

Case I. \( 0 \leq \beta < 2\pi \) and \( 0 \leq N^* \cdot N \). Then \( \int_{\mathcal{V}} \) is empty. Since \( V^n \) is closed, \( V^{n-1} \) is empty. Formula (2.7) then applies, giving

\[ \int_{V^n} (1 - N^* \cdot N) dA = 0. \]

\( dA \) is of fixed sign and \( 1 - N^* \cdot N = 1 - \cos \beta \geq 0 \), where \( \beta \) denotes the angle between \( N \) and \( N^* \). (3.1) then implies \( 1 - N^* \cdot N = 0 \) or \( N^* = N \).

Inasmuch as \( N^* \cdot dX^* = N \cdot dX = 0 \), one has

\[ (N \cdot X) dk = N \cdot (dX^* - kdX) \]

\[ = N^* \cdot dX^* - kN \cdot dX \]

\[ = 0. \]

The set \( S \) of points of \( V^n \) for which \( X \cdot N = 0 \) can have no interior points, since \( S \) would then contain a piece of a hypercone with vertex \( O \). Thus \( V^n - S \) is dense in \( V^n \). Hence \( dk \) is a continuous function on \( V^n \) which vanishes on a dense subset, so \( dk = 0 \) everywhere. Then \( k \) is constant and since \( N^* = N, k \) must be positive.

Case II. \( O \in V^n \) or \( O \in V^{*n} \). One may assume \( O \in V^n \) without loss of generality. Let \( U \) be any open set of \( V^n \) containing \( O \), and let \( V \) be a neighborhood of \( O \) which is contained in \( U \). Let \( V' \) be the boundary of \( V \) (and \( V^n - V \)). Since \( (1 - N^* \cdot N) dA \) does not change sign,

\[ \left| \int_{V^n - U} (1 - N^* \cdot N) dA \right| \leq \left| \int_{V^n - V} (1 - N^* \cdot N) dA \right|, \]

and by (2.7)

\[ \left| \int_{V^n - V} (1 - N^* \cdot N) dA \right| = \frac{1}{n!} \left| \int_{V} (N - N^*) \cdot (X \times dX \times \cdots \times dX) \right|. \]
which can be made small by making $V$ small. The right member of (3.2) is fixed, however, so that

$$\int_{V^n-U} (1 - N^* \cdot N) dA = 0.$$  

As in Case I, $k$ is then a positive constant in $V^n-U$ for any $U$. By continuity, $k$ is then a positive constant throughout $V^n$.

If instead, $M^*X^* = -M_1X$, let $X^* = -kX$ with $k = M_1/M^*$. From (2.7') one has

$$\int_{V^n} (1 + N^* \cdot N) dA = 0,$$

so

$$N^* = -N.$$  

Proceeding as before, $k$ must be a positive constant.

Remark. Consideration of (2.7) and (2.7') gives immediately that one may replace the condition that $F_n$ and $F^*_n$ be closed by: $V^n, V^*_n$ have closed boundaries $V^{n-1}$ and $V^{*n-1}$ such that at corresponding points of the boundaries (a) $N^* = N$ or (b) $N^* = -N$ respectively, and Theorem 1 still holds.

Corollary. Let $V^n$ be a closed orientable hypersurface which is convex with respect to a fixed point $O$. If with $O$ as origin $M_1^*X^* = -M_1X$, where $X'$ denotes the image of $X$ under $f: V^n \rightarrow V^n$, then $V^n$ is symmetric with respect to $O$.

By Theorem 1, $f$ is a homothetic transformation with negative constant of proportionality $-k$ and center $O$. Then $f \circ f = \text{identity}$, so $k^2 = 1$ or $k = 1$.

Proof of Theorem 2. Let $g(X) = (X, f(X)/|f(X)|)$. Since $g$ preserves angles and magnitudes, $g$ is given by a motion (possibly improper) which leaves $O$ fixed. Then $f g^{-1}: V^n \rightarrow g(V^n) \rightarrow V^*_n$ satisfies the conditions of Theorem 1. $f = (f g^{-1}) g$ is then a motion followed by a homothetic transformation with center $O$. Thus $f$ is a similarity with center $O$.

Proof of Theorem 3. Let $g: E^{n+1} - \{O\} \rightarrow E^{n+1} - \{O\}$ denote the inversion with respect to the unit hypersphere. Let $V^n = g(V^n)$. By straightforward calculation, one finds

$$M_1^*X' = -\left( M_1 + 2X \cdot N \right) X.$$
Thus one has (a) $M'_1 X' = M^*_1 X^*$ or (b) $M'_1 X' = -M^*_1 X^*$ respectively. Applying Theorem 1, $f g^{-1}$ is a homothetic transformation with center $O$. $f = (f g^{-1}) g$ is then an inversion with the given properties.

**Remark.** If $V^n$, $V^*$ have closed boundaries $V^{n-1}$ and $V^{*n-1}$ and if at corresponding points of the boundaries one has, respectively, (a) $N^* = -N + 2(X \cdot N/X \cdot X)X$ or (b) $N^* = N - 2(X \cdot N/X \cdot X)X$, the theorem also holds.

Since one has (a) $N^* = N'$ or (b) $N^* = -N'$ respectively, the remark of Theorem 1 gives the desired result.

**Corollary.** If $V^n$ is a closed orientable hypersurface which is convex with respect to a point $O$ not in $V^n$ and if with $O$ as origin,

$$M_1 = -X \cdot N/X \cdot X,$$

then $V^n$ is a hypersphere.

Since $M_1 X = -(M_1 + 2X \cdot N/X \cdot X)X$, each point of $V^n$ is invariant under the inversion in a hypersphere of real radius with center $O$. Thus $V^n$ is this hypersphere.

**References**