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AN UNCOUNTABLE SET OF INCOMPARABLE DEGREES

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The purpose of this note is to prove the following:[1]

Theorem. *There is an uncountable set of pairwise incomparable degrees of recursive unsolvability.*

By Zorn's lemma, there is a maximal set of pairwise incomparable degrees of recursive unsolvability different from $\mathbf{0}$; we must show that this set is not countable. Hence our theorem follows from:

Lemma. *If* $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots$ *is a sequence of degrees different from* $\mathbf{0}$, *then there is a degree* $\boldsymbol{b}$ *which is incomparable with each* $\boldsymbol{a}_n$.

Proof.[2] Let α_n be a function of degree $\boldsymbol{a}_n$; we shall construct a function β of degree $\boldsymbol{b}$. As in [1], β is constructed by defining inductively a function κ such that $\kappa(a) = \bar{\beta}(\nu(a))$ with $\nu(a) = lh(\kappa(a))$; κ and ν must satisfy the conditions that $\kappa(a)$ is a sequence number, $\kappa(a+1)$ extends $\kappa(a)$, and $\nu(a+1) > \nu(a)$. We then have $\beta(a) = (\kappa(a+1))_a - 1$.

Let $\kappa(0) = 1$. To define $\kappa(a+1)$, let $n = (a)_1$ and $e = (a)_2$. If a is even, set

$$\kappa(a + 1) = \kappa(a) \cdot p_{\nu(a)} \exp(\{e\}^{\alpha_n}(\nu(a)) + 2)$$

if $\{e\}^{\alpha_n}(\nu(a))$ is defined, and $\kappa(a+1) = \kappa(a) \cdot p_{\nu(a)}$ otherwise. Then clearly $\beta \neq \{e\}^{\alpha_n}$ for any function β such that $\beta(\nu(a+1)) = \kappa(a+1)$.

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[1] The problem solved in this paper was suggested to the author by C. Spector.

[2] We use the notation of [1] in the proof.

If a is odd, we shall choose $\kappa(a+1)$ so that $\alpha_n \neq \{e\}^\beta$ for any function β such that $\bar{\beta}(\nu(a+1))=\kappa(a+1)$. Since α_n is not recursive, there is a w such that $\alpha_n(w)$ is not equal to

$$U(\mu y[\mathrm{Ext}(y, \kappa(a)) \ \& \ T_1^1(y, e, w)]). \tag{1}$$

If (1) is undefined, i.e., if $(\overline{E}y)(\mathrm{Ext}(y, \kappa(a)) \ \& \ T_1^1(y, e, w))$, then $\{e\}^\beta(w)$ is necessarily undefined, and we may take $\kappa(a+1)=\kappa(a)\cdot p_{\nu(a)}$. Otherwise, there is a y such that $\mathrm{Ext}(y, \kappa(a))$, $T_1^1(y, e, w)$, and $U(y) \neq \alpha_n(w)$. We may then take $\kappa(a+1)$ to be any extension of y of length greater than $\nu(a)$.

Since $\beta \neq \{e\}^{\alpha_n}$ and $\alpha_n \neq \{e\}^\beta$ for all e, $\boldsymbol{b}$ is incomparable with $\boldsymbol{a}_n$.

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