IMMERSION OF MANIFOLDS OF NONPOSITIVE CURVATURE

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In [4] Tompkins proved that a flat compact Riemannian manifold M of dimension n cannot be (isometrically) immersed in Euclidean $(2n-1)$-space. Chern and Kuiper conjectured in [2] that the result holds if the Riemannian (i.e. sectional) curvature K of M is never positive. An algebraic result verifying this conjecture was found by Otsuki [3]. We shall prove:

THEOREM. *Let M be a compact n-dimensional Riemannian manifold and let $\overline{M}$ be a complete simply connected Riemannian manifold of dimension less than 2n. If the Riemannian curvatures K and $\overline{K}$ of M and $\overline{M}$ satisfy $K \leqq \overline{K} \leqq 0$, then M cannot be immersed in $\overline{M}$.*

Simple examples involving spheres and tori show that the theorem fails if either inequality or the simple connectedness of $\overline{M}$ is deleted.

Following [1] we express the second fundamental form information of an immersion $i\colon M \to \overline{M}$ in terms of the *difference transformation* T, a function which assigns to each vector x in M_m (the tangent space to M at $m \in M$) a linear transformation T_x from M_m to the orthogonal complement $M_m^\perp$ of $di(M_m)$ in $\overline{M}_{i(m)}$. The natural definition of T is in terms of the notion of difference of two connections [1], however it may be described in terms of the classical second fundamental form S as follows: S is a function which assigns to each vector $z \in M_m^\perp$ a symmetric linear operator S_z on M_m. If $x \in M_m$ and $z \in M_m^\perp$, let $T_x(z) = S_z(x)$; then uniquely extend T_x to be a skew-symmetric operator on all of $\overline{M}_{i(m)}$. For our purposes, as indicated above, we need only the portion of T_x defined on $di(M_m)$ or, equivalently, M_m. As a function of x, $y \in M_m$, $T_x(y)$ is bilinear and symmetric.

The difference transformation relates the Riemannian curvatures of M and $\overline{M}$ as follows: if x and y span a plane P in M_m, then

$$K(P) = \frac{\langle T_x(x), T_y(y)\rangle - \|T_x(y)\|^2}{\|x \wedge y\|^2} + \overline{K}(di(P)). \tag{1}$$

This formula, the Gauss equation, is readily obtained from the second structural equations of M and $\overline{M}$.

The following lemma extends a well-known Euclidean fact.

LEMMA 1. *Let $i\colon M \to \overline{M}$ be an immersion of a compact Riemannian*

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manifold in a complete simply connected manifold with Riemannian curvature $\bar{K} \leqq 0$. *Then there is a point* $m \in M$ *and a vector* $z \in M_m^\perp$ *such that* $\langle T_x(x), z\rangle < 0$ *for all nonzero* $x \in M_m$.

PROOF. Fix a point $\bar{m}$ in $\bar{M}$ and use the following notation: m, a point of M such that $i(m)$ has maximum distance from $\bar{m}$; σ: $[0, 1] \to \bar{M}$, the unique geodesic from m to $i(m)$; z, the velocity vector $\sigma(1)$ of σ at $i(m)$. If $x \in di(M_m)$ there is a differentiable map r: $[0, 1] \times [0, 1] \to M$ such that (1) $r(\cdot, 0) = \sigma$, (2) for each $v \in [0, 1]$, $r(\cdot, v)$ is a geodesic, (3) $r(0, \cdot) = \bar{m}$ and $r(1, \cdot) \in i(M)$, (4) if X is the vector field on σ such that $X(u)$ is the velocity of $r(u, \cdot)$ at $v = 0$, then $X(1) = x$. Let $l(v)$ be the length of $r(\cdot, v)$, that is, the distance from $\bar{m}$ to $r(1, v) \in i(M)$. Obviously $l'(0) = 0$ and $l''(0) \leqq 0$. The vanishing of the first variation implies $z \in M_m^\perp$. By the Synge formula for the second variation [1] we have

$$sl''(0) = \int_0^1 \{\|X'\|^2 - \bar{K}(\dot{\sigma}, X)\|\dot{\sigma} \wedge X\|\} + \langle T_x(x), z\rangle$$

where s is the length of σ, and X' is the covariant derivative of X. Since $\bar{K} \leqq 0$, the integral term is positive if $x \neq 0$, hence $\langle T_x(x), z\rangle < 0$.

PROOF OF THE THEOREM. Following Tompkins' scheme we reduce the proof to a problem in linear algebra. Let i: $M \to \bar{M}$ be an immersion, where M and $\bar{M}$ are as described in the theorem, except that no restriction is made on the dimension of $\bar{M}$. Let m and z be as in Lemma 1. From the formula (1) and the condition $K \leqq \bar{K}$ we get $\langle T_x(x), T_y(y)\rangle \leqq \|T_x(y)\|^2$ for all $x, y \in M_m$. We need only prove that dimension $M_m^\perp \geqq n$, and this follows from

LEMMA 2. *Let* U *and* V *be finite-dimensional real vector spaces,* V *with an inner product. Suppose that for each* $x \in U$ *there is a linear transformation* T_x: $U \to V$ *such that*:

(1) $\langle T_x(x), T_y(y)\rangle \leqq \|T_x(y)\|^2$ *for all* $x, y \in U$.

(2) $T_x(y)$ *is bilinear and symmetric in* x *and* y.

(3) *There is a vector* $z \in V$ *such that* $\langle T_x(x), z\rangle < 0$ *for all nonzero* x *in* U. *Then dimension* $V \geqq$ *dimension* U.

PROOF. Suppose the contrary; then for every $u \in U$ there is a nonzero $v \in U$ such that $T_u(v) = 0$, hence $\langle T_u(u), T_v(v)\rangle \leqq 0$. Consider the real-valued function f on $U - \{0\}$ for which

$$f(u) = \frac{\langle T_u(u), z\rangle}{\|T_u(u)\| \, \|z\|}.$$

Since f is continuous and constant on lines through the origin, it has a minimum, say $f(x)$. Let y be a nonzero vector such that $T_x(y)=0$ and $\langle T_x(x), T_y(y)\rangle \leqq 0$. Note that $T_x(x)$ and $T_y(y)$ are independent, for otherwise we may assume $T_x(x)+T_y(y)=0$, which contradicts (3). Let P be the plane spanned by these two vectors, and let $z^\perp$ be the set of vectors orthogonal to z. Since $T_x(x)$ is not in $z^\perp$, the subspace $P\cap z^\perp$ is 1-dimensional. Let p be the unique unit vector in P such that $\langle p, z\rangle < 0$ and p is orthogonal to $P\cap z^\perp$. Clearly $\langle p, z\rangle < \langle q, z\rangle$ if q is a unit vector in P different from p. Now the definition of p together with the inequalities $\langle T_x(x), T_y(y)\rangle \leqq 0$, $\langle T_x(x), z\rangle < 0$, $\langle T_y(y), z\rangle < 0$ imply that p lies between $T_x(x)$ and $T_y(y)$, that is, that we may write $p=\lambda^2 T_x(x)+\mu^2 T_y(y)$. Thus $p=T_{\lambda x+\mu y}(\lambda x+\mu y)$. In view of the minimality properties of p and $T_x(x)$ we must have $T_x(x)$ a positive scalar multiple of p. But since p is orthogonal to $P\cap z^\perp$, this implies $\langle T_y(y), z\rangle \geqq 0$, a contradiction.

It is clear that the theorem holds in slightly more general form, patterned on the full Chern-Kuiper conjecture, namely: if there is at each point m of M a q-dimensional subspace S_m $(q\geqq 2)$ of M_m such that $K\leqq \bar{K}$ holds when K is restricted to planes in any S_m, then immersion is impossible if the dimension of $\bar{M}$ is less than $n+q$.

References

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