IMBEDDING DECOMPOSITIONS OF E^3 IN E^4

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1. Introduction. We consider monotone decompositions of E^3 and study the possibility of imbedding the decomposition spaces in E^4. Attention has recently been called [1; 2] to the fact that certain useful decompositions of E^3 can be imbedded in E^4. It would be of interest to find a simple decomposition of E^3 that could not be so imbedded.

Our first observation is that if the decomposition contains only two nondegenerate elements, then such an imbedding is always possible. However, if the nondegenerate elements of the decomposition consist of nine tamely imbedded (actually planar) circles, appropriately linked, then such an imbedding is not possible. We conjectured that the same was true in case the nondegenerate elements were three planar circles each pair of which was linked, but a neat proof by Goblirsch [4] shows our intuition was faulty.

THEOREM 1. *If a monotone decomposition of E^3 contains only two nondegenerate elements, then the decomposition space is imbeddable in E^4.*

PROOF. Let C_1 and C_2 be the nondegenerate elements and consider E^3 to be the x_1, x_2, x_3 plane in E^4. Let P_1 and P_2 be points in E^4 with x_4 coordinates 1 and -1 respectively. Let B_i be the cone over C_i from P_i, $i=1, 2$. Let U_1 and U_2 be disjoint neighborhoods of C_1 and C_2 respectively, and let f_i be a function mapping E^3 onto the unit interval such that the complement of U_i is sent to 0 and precisely all points of C_i are sent to 1, for $i=1, 2$.

We map the E^3 plane into E^4 as follows. A point $x=(x_1, x_2, x_3, x_4)$ in U_i goes to the point on the segment from x to P_i whose fourth coordinate is $(-1)^i f_i(x)$. Points not in $U_1 \cup U_2$ are left fixed. It is clear that the image under this map is just the decomposition space, and it is in E^4. This proves the theorem.

Theorem 1 has the following obvious generalization.

THEOREM 2. *Let $C_1, \cdots, C_n$ be the nondegenerate elements of a monotone decomposition of E^3, and consider E^3 to be a 3-plane in E^4. If there exist points $P_1, \cdots, P_n$ in $E^4 - E^3$ such that the cones B_i (B_i being the*

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cone over C_i from P_i) are all disjoint, then the decomposition space of E^3 can be imbedded in E^4.

As an application we note that if the nondegenerate elements of the decomposition consist of the three "Ballantine Ale" circles, then the decomposition space can be imbedded in E^4.

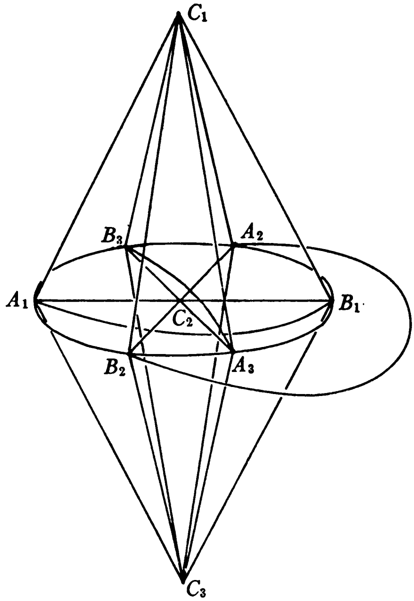

FIG. 1

In order to show that certain decomposition spaces of E^3 cannot be imbedded in E^4 we will show that they contain certain 2-polyhedra which cannot be imbedded in E^4. The example we use is a generalization of the "utility," or "cranky neighbors" puzzle where there are six points A_1, A_2, A_3, B_1, B_2, B_3 in the plane and the problem is to join each of the A's to each of the B's so that no two of the joining arcs intersect except possibly in an end point of each. The generalization of this skew curve of type 1 in the next dimension is the 2-complex P with nine vertices A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 and twenty-seven 2-simplexes of the sort $A_iB_jC_k$. Flores showed [3] that this complex cannot be imbedded in E^4. (An independent proof of the fact that P cannot be imbedded in E^4 can be obtained using the work of Shapiro [5] as indicated in §4.) In the next section we show that a certain decomposition of E^4 whose nondegenerate elements are nine circles cannot be imbedded in E^4 because it contains the polyhedron P.

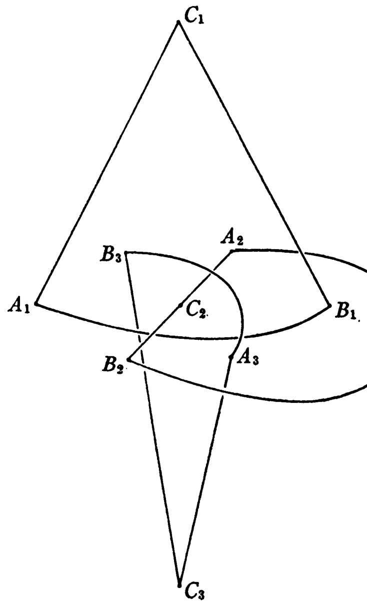

FIG. 2

2. **A construction.** We take nine points in E^3 as shown in Figure 1, and note that all of the "triangles" of the type $A_iB_jC_k$ except three can be filled in in E^3. If all such triangles are added (not in E^3, but say in E^5) the result is the Flores polyhedron P. We fill in all except the three $A_iB_iC_i$ triangles (see Figure 2), and denote the resulting polyhedron by Q.

Let J_i be the circle $A_iB_iC_i$ for $i=1, 2, 3$. The polyhedron R obtained from Q by shrinking the three J_i to points will be in the decomposition space of E^3 resulting from the decomposition having the J_i as the only nondegenerate elements. However, as we show in the next section, R can be imbedded in E^4. Goblirsch's result [5] gives an independent proof of this imbeddability. However, if we put in some more circles to shrink, we can obtain the Flores polyhedron P.

Figure 3 shows the circle $A_1B_1C_1$ with a strip S joined to it. To prevent S intersecting other 2-simplexes we cut small open disks (with boundaries β_1, β_2, β_3) out of the 2-simplexes of $A_3B_3C_1$, $A_2B_1C_2$, $A_1B_3C_2$ respectively. If the four circles β_1, β_2, β_3, β_4 are all shrunk to points, S becomes a disk as does each of the 2-simplexes from which we have removed small open disks. Hence, if we put in four such circles for each of $A_2B_2C_2$ and $A_3B_3C_3$, we get twelve circles, and when they are shrunk the resulting 2-polyhedron is the Flores polyhedron P. Thus we have proved that there exists twelve planar circles in E^3

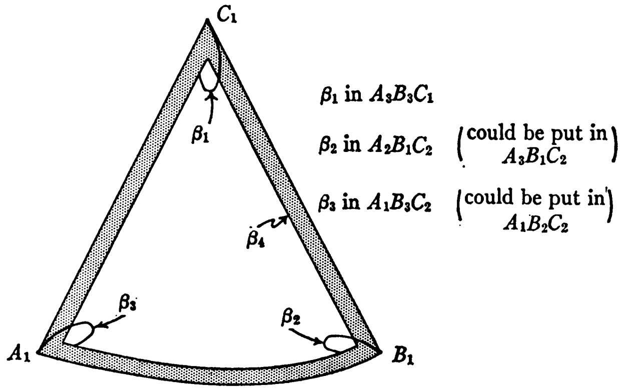

FIG. 3

such that if each is shrunk to a point, the resulting space cannot be imbedded in E^4.

When Ronald H. Rosen saw the preceding example, he suggested the following improvement of it. If we permit the curve β_4 to touch $A_1B_1C_1$ at C_1, S would no longer be an annulus but would be topologically equivalent to a disk with a circular hole in it such that the boundary of the hole is tangent to the boundary of the disk. However, S would become a disk if β_4 were shrunk to a point. An advantage in using the new S rather than the old is that no hole bounded by β_1 need be cut in $A_3B_3C_1$. Hence, this example shows the following.

THEOREM 3. *There exist nine planar circles in E^3 such that if each is shrunk to a point, then the resulting space cannot be imbedded in E^4.*

The nine circles referred to in Theorem 3 consist of three linking curves similar to those shown in Figure 2 together with two small circles linking each of these large ones. Two of the larger curves can be joined by a straight line interval so that the eight sets consisting of this sum and the other seven circles satisfy the hypothesis of Theorem 2. Hence, we have the following result.

THEOREM 4. *If either a 1-simplex or a 2-simplex of the Flores polyhedron P is shrunk to a point, the resulting set can be imbedded in E^4.*

3. **Imbedding R in E^4.** Returning to Figure 1, we consider the polyhedron R obtained from Q by shrinking J_1, J_2, J_3 to points. Part of the resulting polyhedron is shown in Figure 4 in which we let $V_i = A_i = B_i = C_i$. The part T of R shown in Figure 4 is obtained from R by omitting the six 2-simplexes with vertices V_1, V_2, V_3 and it consists of three 2-spheres joined at poles so as to form a loop.

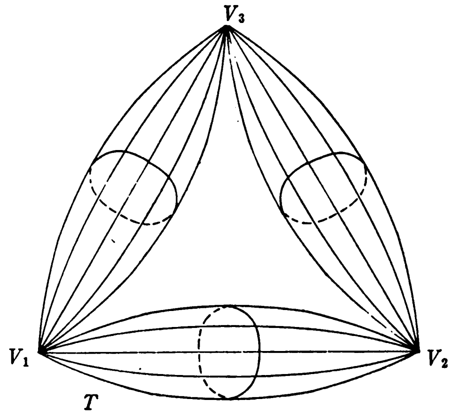

FIG. 4

Each 2-sphere is divided into six 2-cells corresponding to six 2-simplexes in Q.

Now we consider T in E^3 with E^3 in E^4, and we assert that the remaining six simplexes of R can be added to T in E^4. This is demonstrated by using the chart in Figure 5 in which the vertices have been "spread out." The six simplexes to be added to T are indicated by S_1, S_2, S_3, S_4, S_5, S_6 in Figure 5. We see in Figure 5 that S_4, S_2 and S_5 are nonintersecting. This implies that these three 2-simplexes can be added to T in E^3. Of the remaining three, S_1 and S_3 do not intersect. Hence we can add the two simplexes for these in one domain of E^4-E^3. We add S_6 in the other domain and the imbedding is completed.

Questions. We list two questions raised by this paper.

1. If three mutually exclusive bounded continua in E^3 are shrunk to points, can the resulting space be imbedded in E^4? Theorem 1 gives an answer in the case of two continua and Goblirsch's result gives an answer in case the three continua are simple closed curves appropriately linked.

2. What is the minimum integer n such that there are n mutually exclusive circles in E^3 such that if they are shrunk to points, the decomposition space cannot be imbedded in E^4? There is a gap between the integer 9 of Theorem 3 and the integer 3 in Goblirsch's result.

4. **Nonimbeddability of P in E^4.** Let the vertices of P be placed in general position in E^4. This will define a map $f: P \rightarrow E^4$ by extending linearly, and only pairs of images of 2-simplexes can have nonempty intersection. Let $\overline{P}^*$ denote the deleted symmetric product of P [**5**, p. 257]. The cocycle which assigns to the 4-cell $\alpha \times \beta$ of $\overline{P}^*$ the

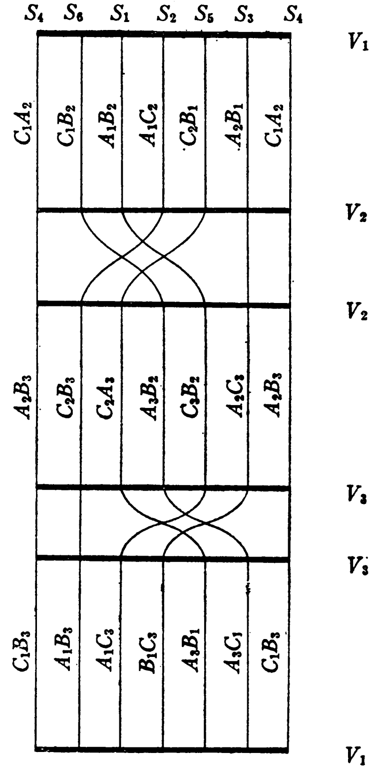

FIG. 5

intersection number of $f(\alpha)$ and $f(\beta)$ is denoted by $m_f(P)$. Its cohomology class is independent of f [5, p. 259]. Hence, if P can be imbedded in E^4, then for any f we must have that $m_f^4(P)$ is cohomologous to zero. We will choose a specific f, assume that $m_f^4(P)$ is a coboundary, and obtain a contradiction.

We place the vertices of P along a G-curve [5, p. 259] in the following order $A_1B_1C_1A_2B_2C_2A_3B_3C_3$ and extend linearly to obtain f. Now the images of two 2-simplexes (having no common vertex) will intersect if and only if their vertices alternate in the above order. If the images of α and β intersect, then $m_f^4(\alpha\times\beta)=1$ and otherwise $m_f^4(\alpha\times\beta)=0$. For P we obtain nine values one and ninety nine zero for the 108 4-cells in $\overline{P}^*$. Suppose C is a 3-cochain such that $\delta C=m_f^4(P)$. Then C can be considered to be an integral linear combination of the 324 3-cells of $\overline{P}^*$, and the equation $\delta C=m_f^4(P)$ becomes a system of 108 equations in 324 unknowns. It is a routine (but tedious) matter to show that these equations have no integral solution, so that P cannot be imbedded in E^4.

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