INTRINSIC CHARACTERIZATIONS OF SOME ADDITIVE FUNCTORS

CHARLES E. WATTS

Our purpose here is to obtain intrinsic functorial characterizations of the functors Hom and $\otimes$ and thus to account in part for the distinguished role played by then in homological algebra. In all that follows, Λ, Γ are rings with unit, Z the ring of integers. The category of all Γ-Λ-bimodules with Γ operating on the left, Λ on the right, is denoted by ${}_\Gamma\mathfrak{M}_\Lambda$, the category of left Λ-modules (Γ-modules) by ${}_\Lambda\mathfrak{M}({}_\Gamma\mathfrak{M})$, etc. All functors are assumed additive. We use throughout the terminology of [1].

THEOREM 1. *Let T be a right-exact covariant functor on ${}_\Lambda\mathfrak{M}$ to ${}_\Gamma\mathfrak{M}$ which commutes with direct sums (i.e., is of type $L\Sigma$). Then there is an object C in ${}_\Gamma\mathfrak{M}_\Lambda$ and a natural equivalence of functors ψ: $C\otimes_\Lambda \approx T$.*

PROOF. Given an object A of ${}_\Lambda\mathfrak{M}$ and $a\in A$, define $\phi_a:\Lambda\to A$ by $\phi_a(\lambda)=\lambda a$. Then $T\phi_a$: $T\Lambda\to TA$ and we define a function ψ_0^A: $T\Lambda\times A\to TA$ by $\psi_0^A(\bar\lambda, a)=T\phi_a(\bar\lambda)$, $\bar\lambda\in T\Lambda$, $a\in A$. It is easily checked that ψ_0^Λ: $T\Lambda\times\Lambda\to T\Lambda$ gives $T\Lambda$ the structure of a right Λ-module, compatible with its left Γ-module structure. Thus we have made $T\Lambda$ into an object of ${}_\Gamma\mathfrak{M}_\Lambda$; $T\Lambda$, considered as an object of ${}_\Gamma\mathfrak{M}_\Lambda$, will be denoted by C.

Now for any A, ψ_0^A: $C\times A\to TA$ and a simple computation shows that ψ_0^A is bilinear. Hence ψ_0^A can be lifted uniquely to a Γ-homomorphism ψ^A: $C\otimes_\Lambda A\to TA$, and the maps ψ^A form a natural transformation of functors. Moreover, ψ^Λ is the natural isomorphism of $C\otimes_\Lambda\Lambda$ onto $T\Lambda$. Since T and $C\otimes_\Lambda$ both commute with direct sums, it follows that ψ^F is an isomorphism whenever F is a *free* Λ-module.

Finally, let A be any Λ-module; choose an exact sequence

$$0\to R\to F\to A\to 0$$

with F free. Since T and $C\otimes_\Lambda$ are right-exact, we have an induced commutative diagram with exact rows

$$\begin{array}{ccccccc} C\otimes R & \to & C\otimes F & \to & C\otimes A & \to & 0 \\ \downarrow\psi^R & & \downarrow\psi^F & & \downarrow\psi^A & & \\ TR & \to & TF & \to & TA & \to & 0 \end{array}$$

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from which it is apparent that ψ^A is an epimorphism. Since A was arbitrary, ψ^R is an epimorphism. An easy chase about the diagram then reveals that ψ^A is a monomorphism, q.e.d.

Now suppose Λ is left-Noetherian and let ${}_\Lambda\overline{\mathfrak{M}}$ be the category of finitely-generated left Λ-modules. Then we can modify the latter part of the above proof by choosing the sequence $0 \rightarrow R \rightarrow F \rightarrow A \rightarrow 0$ so that F, and therefore R, is finitely generated. Since every functor commutes with *finite* direct sums, we get

THEOREM 2. *If Λ is left-Noetherian and if T is any right-exact covariant functor on ${}_\Lambda\overline{\mathfrak{M}}$ to ${}_\Gamma\mathfrak{M}$, then there is an object C of ${}_\Gamma\mathfrak{M}_\Lambda$ and a natural equivalence of functors ψ: $C \otimes_\Lambda \approx T$.*

For example, if Λ is left-Noetherian and A is a left Λ-module with $\dim A \leqq n$, then $\operatorname{Ext}^n_\Lambda(A, B) \approx \operatorname{Ext}^n_\Lambda(A, \Lambda) \otimes_\Lambda B$ for each finitely-generated B.

THEOREM 3. *Let T be any left-exact contravariant functor on ${}_\Lambda\mathfrak{M}$ to ${}_\Gamma\mathfrak{M}$ which converts direct sums into direct products (i.e., is of type $R\Pi$). Then there is a left Λ-Γ-bimodule C and a natural equivalence of functors ψ: $T \approx \operatorname{Hom}_\Lambda(\ , C)$.*

The proof of Theorem 3 is completely analogous to that of Theorem 1, and is omitted. It can be varied to provide an analogue of Theorem 2. These theorems are crucially dependent on the possibility of representing an arbitrary module as a quotient module of a direct sum of copies of Λ. To characterize the Hom functor in the covariant variable, we first establish the existence of a construction dual to this.

DEFINITIONS. If A is a module and X any set, the module A^X is defined to be the direct product of copies of A indexed by X, or equivalently the set of all functions from X to A under pointwise addition and scalar multiplication. If A and V are two left modules, the *evaluation map*[1] for A and V is the homomorphism

$$\alpha: A \longrightarrow V^{\operatorname{Hom}(A,V)}$$

given by $\alpha a(f) = f(a)$. A left module V is *distinguishing* provided (i) V is injective and (ii) for each left module A, the evaluation map for A and V is a monomorphism. (Condition (ii) can be restated: given any nonzero element a of a left module A, there exists f: $A \rightarrow V$ such that $f(a) \neq 0$.)

LEMMA 4. *For every ring Λ, there exists a distinguishing left Λ-module.*

PROOF. Let V' be the direct product of all the modules Λ/I, I a left

[1] This terminology is due to J. L. Kelley.

ideal of Λ. Let V be injective, $V \supset V'$. If $0 \neq a \in A$, there is a natural map $f': \Lambda a \rightarrow \Lambda / I_a$ such that $f'a \neq 0$, where Λa is the submodule of A generated by a and I_a is the annihilator ideal of a. Since V is injective, f' can be extended to a map $f: A \rightarrow V$ with $fa \neq 0$.

We remark that the above construction is highly inefficient; if $\Lambda = Z$, then we can take for V the group of rationals mod one.

LEMMA 5. *Let T be a functor from ${}_\Lambda\mathfrak{M}$ to ${}_\Gamma\mathfrak{M}$ which commutes with inverse limits (i.e., is of type $R\Pi^*$). Then T commutes with direct products.*

For an arbitrary direct product is the inverse limit of finite direct products, and every additive functor commutes with finite direct products.

THEOREM 6. *Let T be a covariant left-exact functor from ${}_\Lambda\mathfrak{M}$ to ${}_Z\mathfrak{M}$ which commutes with inverse limits. Then there exists a left Λ-module C and a natural equivalence of functors ψ: $\mathrm{Hom}_\Lambda(C, \ ) \approx T$.*

PROOF. By Lemma 4, we choose a distinguishing left Λ-module V and set $C' = V^{TV}$. Let $e \in TV^{TV}$ be the identity function; we regard e as a member of TC' by means of Lemma 5.

For each left Λ-module A, we define η_A: $\mathrm{Hom}(C', A) \rightarrow TA$ by

$$\eta_A(f) = Tf(e), \qquad f \in \mathrm{Hom}(C', A).$$

The maps η_A yield a natural transformation $\mathrm{Hom}(C', \ ) \rightarrow T$. We claim that η_V maps $\mathrm{Hom}(C', V)$ *onto* TV. For if $v \in TV$ and f is the vth coordinate projection of $C' = V^{TV}$ onto V, then $\eta_V(f) = Tf(e) = v$, because T is of type $R\Pi$ by Lemma 5.

We next describe the kernel of η_V. If M is any submodule of C', we shall identify TM with the submodule $\mathrm{Im}\ Ti$ of TC', where i: $M \rightarrow C'$ is the inclusion; this is possible because T is left-exact. Let C be the intersection of all submodules M of C' such that $e \in TM$. Now the family of such submodules M together with all their inclusion maps make up an inverse limit system, whose inverse limit is C. Since T commutes with inverse limits and preserves inclusions, it follows that TC is the intersection of the submodules TM of TC'. Hence, in particular, $e \in TC$. It follows that if $\eta_V(f) = Tf(e) = 0$, then $\mathrm{Ker}\, f \supset C$, and conversely, if $\mathrm{Ker}\, f \supset C$, then $e \in T\ \mathrm{Ker}\, f = \mathrm{Ker}\ Tf$, so $\eta_V(f) = 0$. Thus $\mathrm{Ker}\ \eta_V \approx \mathrm{Hom}(C'/C, V)$.

Since $e \in TC$, we can define a natural transformation ψ: $\mathrm{Hom}_\Lambda (C, \ ) \rightarrow T$ by

$$\psi_A(f) = Tf(e), \qquad f \in \mathrm{Hom}(C, A).$$

The exact sequence $0 \to C \to C' \to C'/C \to 0$ induces an exact sequence

$$0 \to \mathrm{Hom}(C'/C, V) \to \mathrm{Hom}(C', V) \to \mathrm{Hom}(C, V) \to 0$$

because V is injective. On the other hand, we have shown that the sequence

$$0 \to \mathrm{Hom}(C'/C, V) \to \mathrm{Hom}(C', V) \xrightarrow{\eta_V} TV \to 0$$

is exact. Hence $\mathrm{Hom}(C, V) \approx TV$ and it is easy to see that ψ_V is the natural isomorphism arising from the last two sequences.

From here on, the proof proceeds much like that of Theorem 1. Let A be any left Λ-module. The exact sequence

$$0 \to A \xrightarrow{\alpha} V^{\mathrm{Hom}(A,V)} \to \mathrm{Coker}\, \alpha \to 0,$$

where α is the evaluation map, yields a commutative diagram

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}(C, A) & \to & \mathrm{Hom}(C, V)^{\mathrm{Hom}(A,V)} & \to & \mathrm{Hom}(C, \mathrm{Coker}\, \alpha) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & TA & \longrightarrow & TV^{\mathrm{Hom}(A,V)} & \longrightarrow & T\, \mathrm{Coker}\, \alpha
\end{array}$$

in which the central vertical arrow is an isomorphism. We prove as before that each ψ_A is an isomorphism, so that ψ is in fact a natural equivalence, q.e.d.

Reference

1. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, 1956.

The University of Chicago