ON THE PRIMITIVITY OF HOPF ALGEBRAS OVER A FIELD WITH PRIME CHARACTERISTIC

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We recall that an \( H \)-space consists of a topological space \( T \) with a base point \( e \in T \) and a (continuous) map \( \nabla: T \times T \to T \) such that \( \nabla i \approx I \) and \( \nabla j \approx I \), where \( i \) and \( j \) are defined by \( i(t) = (t, e) \) and \( j(t) = (e, t) \), \( I \) is the identity map of \( T \), and "\( \approx \)" means "homotopic relative to \( e \)." The multiplication \( \nabla \) is homotopy-associative if
\[
\nabla(\nabla \times I) \approx \nabla(I \times \nabla);
\]
it is homotopy-commutative if
\[
\rho \nabla \approx \nabla
\]
where \( \rho \) is defined by \( \rho(s, t) = (t, s) \), \( (s, t) \in T \). We shall assume throughout that \( T \) is arcwise connected.

Let \( \mathcal{H} \) be an associative and anticommutative graded \( K \)-algebra with unit 1, where \( K \) is a field. We assume throughout that \( \mathcal{H}^i = 0 \) if \( i < 0 \), and \( \mathcal{H}^0 = K \cdot 1 \). Let \( \mathcal{H}^+ \) denote the submodule spanned by the elements of positive degree. \( \mathcal{H} \) is a Hopf algebra over \( K \) if there is an algebra homomorphism \( \Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) (regarding \( \mathcal{H} \otimes \mathcal{H} \) as a graded \( K \)-algebra in the usual way) such that
\[
\Delta''(x) = \Delta(x) - \Delta'(x) \in \mathcal{H}^+ \otimes \mathcal{H}^+,
\]
where \( \Delta': \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) is defined by
\[
\Delta'(1) = 1 \otimes 1, \quad \Delta'(x) = x \otimes 1 + 1 \otimes x, \quad x \in \mathcal{H}^+.
\]
We shall refer to \( \Delta \) as the coproduct.

The coproduct is associative if
\[
(\Delta \otimes I) \Delta = (I \otimes \Delta) \Delta
\]
where \( I \) is the identity map of \( \mathcal{H} \); it is anticommutative if
\[
\theta \Delta = \Delta,
\]
where \( \theta \) is defined by
\[
\theta(x \otimes y) = (-1)^{ij} y \otimes x, \quad x \in \mathcal{H}^i, \ y \in \mathcal{H}^j.
\]

By a Hopf subalgebra we mean a graded subalgebra \( G \) such that \( \Delta(G) \subseteq G \otimes G \).

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It is well-known that the cohomology algebra \( H^*(T, K) \), where \( T \) is an \( H \)-space and \( K \) is a field, is a Hopf algebra with coproduct \( \Delta = \nabla^* \) (assuming the usual identification given by the Künneth formula). Comparing (1.1) and (1.3), evidently homotopy-associativity of \( \nabla \) implies associativity of \( \Delta \). It is known that \( \theta = \rho^* \); hence, comparing (1.2) and (1.4), we see that homotopy-commutativity of \( \nabla \) implies anticommutativity of \( \Delta \).

Let \( (H, \Delta) \) be a Hopf algebra over \( K \). An element \( y \in H \) is primitive if \( \Delta''(y) = 0 \). Let \( \pi \subseteq H \) be the subalgebra generated by the primitive elements. It is easy to see that \( \pi \) is a Hopf subalgebra. If \( \pi = H \) then we call \( H \) a primitive Hopf algebra. The following theorem was proved by the author [3, Theorem 2.10], and independently by J. C. Moore [5].

(1.5) If \( H \) is a Hopf algebra over a field of characteristic zero and the coproduct is associative and anticommutative then \( H \) is primitive.

A simple algebraic example shows that (1.5) is not true in general if the field has prime characteristic. The following theorem is due to H. Samelson [6] and J. Leray [4]:

(1.6) Let \( H \) be a Hopf algebra over a field and let the coproduct be associative. If \( H \) is an exterior algebra generated by odd degree elements then it is primitive.

The purpose of this paper is to establish primitivity of \( H^*(T, Z_p) \) for some \( H \)-spaces \( T \), where \( Z_p \) is the ring integers modulo a prime \( p \). We shall make use of properties of the Steenrod cohomology operations [7] which we denote by

\[
St_p^i = \begin{cases} 
S_{p^i}^i & \text{(squares)}, \\
\left(p^i\right)_p & \text{(reduced powers)},
\end{cases}
\]

if \( p = 2 \), if \( p > 2 \).

Let \( T \) be an \( H \)-space and suppose \( H^*(T, Z_p) \) is a polynomial ring \( Z_p[X] \) where \( X \subseteq H^*(T, Z_p) \) and consists of even degree elements if \( p \neq 2 \). The operations \( St_p^i \) are said to split on \( X \) if for all \( i \geq 0 \) and \( x \in X \), \( St_p^i(x) \) is in the subalgebra generated by \( x \).

**Theorem 1.** Let \( T \) be an arcwise connected \( H \)-space with homotopy-associative and homotopy-commutative multiplication. If \( H^*(T, Z_p) = Z_p[X] \) and the Steenrod cohomology operations split on \( X \) then \( H^*(T, Z_p) \) is primitive.

We remark that it then follows on using a Künneth formula that \( H^*(T, K) \) is primitive if \( K \) is a field of characteristic \( p \).

As an application of Theorem 1 we shall prove the following theorem. For a fixed prime \( p \), a topological space \( E \) is \( p \)-elementary if
$H^*(E, \mathbb{Z}_p) \cong \mathbb{Z}_p$ or $H^*(E, \mathbb{Z}_p) = \mathbb{Z}_p[x]$. As examples we cite: The real projective plane ($p \neq 2$), the loop spaces $\Omega(S^{2n+1})$ and complex projective space of infinite dimensions. On the other hand the loop spaces $\Omega(S^{2n})$ are not $p$-elementary for any $p$.

**Theorem 2.** Let $p$ be a fixed prime and $E = E_1 \times \cdots \times E_n$, where the $E_i$ are $p$-elementary spaces. Let $T$ be an arcwise connected $H$-space with homotopy-associative and homotopy-commutative multiplication. If there is a map $f: T \to E$ or $f: E \to T$ which induces an isomorphism of the cohomology algebras with coefficients in $\mathbb{Z}_p$, then $H^*(T, \mathbb{Z}_p)$ is primitive.

**Proof.** Using the Künneth formula and $f^*$ we may represent

$$H^*(T, \mathbb{Z}_p) = \mathbb{Z}_p[x_1, x_2, \cdots, x_n]$$

where $x_i$ generates $H^*(E_i, \mathbb{Z}_p)$ (we may ignore the trivial factors). In view of the Cartan tensor product formula (see [2, Exposé 16, bis 1]) the $S^p_t$ split on $H^*(E_1 \times \cdots \times E_n, \mathbb{Z}_p)$ and hence on $\{x_1, x_2, \cdots, x_n\}$ since they commute with $f^*$. Thus the theorem follows from Theorem 1.

**Corollary.** A necessary condition that an arcwise connected $H$-space $T$ with homotopy-associative and homotopy-commutative multiplication be homotopically equivalent to a cartesian product of $p$-elementary spaces $E_1, E_2, \cdots, E_n$ is that $H^*(T, \mathbb{Z}_p)$ be primitive.

**Remark.** Even if the $E_i$ are all $\pi$-spaces, the map $f$ is not required to commute with the multiplication in $T$ and the induced multiplication in $E_1 \times E_2 \times \cdots \times E_n$.

2. **The main lemma.** Let $H = K[X]$ be a Hopf algebra over a field $K$ of prime characteristic $p$. We note:

(2.1) If $p \neq 2$ each $x \in X$ has even degree.

(2.2) We may assume that each $x \in X \cap \pi$ is primitive.

The first is a well-known consequence of the theorem of A. Borel (see [1, Théorème 6.1]). The second follows from Theorem 2.7 in [3].

We shall assume throughout that $X$ is well-ordered in such a way that if $x$ has lower degree than $y$ then $x < y$. By a normal monomial we shall mean a product of the form $M = x_1^{m_1}x_2^{m_2} \cdots x_t^{m_t}$, where the $x_i \in X$ and $x_i < x_{i+1}$. We call $m_1 + \cdots + m_t$ the length of $M$ and the number of positive exponents its width. If $R$ and $S$ are normal monomials their juxtaposition $RS$ (corresponding to their product as elements of $H$) is equal to a unique normal monomial which we denote by $\nu(RS)$. If
By induction on width one proves readily

(2.3) If $M$ is a normal monomial whose factors are primitive then

$$\Delta(M) = \sum_{\nu(RS) = M} [R, S] R \otimes S,$$

where the summation extends over distinct pairs of normal monomials $R, S$.

Let $z \in X$ be such that $\Delta''(z) \in \pi \otimes \pi$. Then we may write (uniquely)

$$\Delta''(z) = \sum a(M, N) M \otimes N, \quad a(M, N) \in K,$$

where the summation extends over finitely many distinct pairs of normal monomials $M, N$ in primitive elements of $X$ and the degree of $MN$ is equal to the degree of $z$. The proof of Theorem 1 will depend on the

MAIN LEMMA. Let $H$ be a Hopf algebra with an associative and anticommutative coproduct $\Delta$ over a field $K$ with prime characteristic $p$. Let $H = K[x]$, where $X \subset H$, and let $z \in X$ be such that $\Delta''(z) \in \pi \otimes \pi$. Then there is an element $v \in H$ with the same degree as $z$ such that $z - v \in \pi$ and

$$\Delta''(v) = \sum \sum a(x^m, x^n) x^m \otimes x^n, \quad a(x^m, x^n) \in K$$

where the outer summation is over (primitive) $x \in X$ and the inner summation is over (positive) $m$ and $n$ with $m + n$ a power of $p$.

We shall first prove some subsidiary lemmas. It will be convenient to extend the definition of $a(M, N)$ in (2.4) as follows: If $Q, R, S, T$ are normal monomials in primitive elements of $X$ then

$$a(QR, ST) = a(v(QR), v(ST)).$$

LEMMA 2.1. If $R, S, T$ are normal monomials with $R \neq 1$ and $T \neq 1$ then

$$a(RS, T)[R, S] = a(R, ST)[S, T].$$

PROOF. Since the coproduct is associative we may equate the coefficients of $R \otimes S \otimes T$ in $(\Delta \otimes I)\Delta(z)$ and $(I \otimes \Delta)\Delta(z)$. Since $R \neq 1$ and $T \neq 1$, it is readily seen $(\Delta \otimes I)\Delta'(z)$ and $(I \otimes \Delta)\Delta'(z)$ contribute nothing to these coefficients. We have
\[(2.6) \quad (\Delta \otimes I)\Delta''(z) = \sum a(M, N)\Delta(M) \otimes N,\]
\[(2.6)' \quad (I \otimes \Delta)\Delta''(z) = \sum a(M, N)M \otimes \Delta(N).\]

Now using (2.3) it follows that the coefficients of \(R \otimes S \otimes T\) in (2.6) and (2.6)', respectively, are
\[a(v(RS), T)[R, S] = a(RS, T)[R, S],\]
\[a(R, v(ST))[S, T] = a(R, ST)[S, T],\]
and the lemma is proved.

**Lemma 2.2.** If \(v(MN)\) is of the form \(w^kQ\), where \(k \geq 1\) is the multiplicity of \(w\) and \(Q \neq 1\) is normal then
\[a(M, N) = [M, N]a(w^k, Q).\]

**Proof.** It suffices to consider \(M\) and \(N\) as normal monomials of the form \(x^mR\) and \(y^nS\) respectively, where \(m\) and \(n\) are the corresponding (positive) multiplicities of \(x\) and \(y\) and \(R\) and \(S\) are normal. Note that if \(R = 1\) and \(S = 1\) the lemma follows at once from anticommutativity of \(\Delta\). Assume that not both \(R = 1\) and \(S = 1\); we consider 3 cases:

(i) \(x < y\). If \(R = 1\) the lemma is trivial. If \(R \neq 1\) then
\[a(x^mR, N) = [R, N]a(x^m, RN) = [R, N]a(x^m, Q) = [M, N]a(x^m, Q).\]

(ii) \(x = y\). If \(R = 1\) then
\[a(x^m, x^nS) = [x^m, x^n]a(x^{m+n}, Q) = [x^m, N]a(x^{m+n}, Q).\]
If \(R \neq 1\) then
\[a(x^mR, x^nS) = [R, N]a(x^m, Rx^nS)\]
\[= [R, N]a(x^m, x^nRS)\]
\[= [R, N][x^m, x^n]a(x^{m+n}, RS)\]
\[= [R, N][x^m, x^n]a(x^{m+n}, Q)\]
\[= [x^mR, N]a(x^{m+n}, Q).\]

(iii) \(x > y\). Using case (i) we may write
\[a(N, M) = [N, M]a(y^k, Q).\]
Note that \(a(M, N) = a(N, M)\) by anticommutativity of \(\Delta\).

**Lemma 2.3.** If \(x \in X\), \(m+n=r+s\), and
\[(2.7) \quad (s - n, n) \not\equiv 0 \pmod{p} \quad s \geq n,\]
then
\[(r, s)a(x^m, x^n) = (m, n)a(x^r, x^s).\]

**Proof.** By Lemma 2.1,
\[(r, m - r)a(x^m, x^n) = (s - n, n)a(x^r, x^s).\]

If we multiply by \((m, n)\) and use the identity
\[(m, n)(r, m - r) = (r, s)(s - n, n),\]
we get
\[(r, s)(s - n, n)a(x^m, y^n) = (m, n)(s - n, n)a(x^r, x^s).\]

In view of (2.7) we may divide out \((s - n, n)\).

**Lemma 2.4.** If \(x \in X\) and \(m + n = q\rho^i\), where \(q > 1\) and \(q \not\equiv 0 \pmod{p}\) then

(2.8) \(a(x^m, x^n) = 0\) if \(\rho^i\) does not divide \(m\) and \(n\),

(2.9) \(a(x^{\rho^i}, x^{\rho^i}) = (r, s)a(x^{(q - 1)p^i}, x^{q^i})/q\) if \(s \not\equiv 0 \pmod{p}\).

**Proof of (2.8).** Suppose \(n < (q - 1)p^i\). Since \(n\) is not divisible by \(\rho^i\),
\[(q - 1)p^i - n, n) \equiv 0 \pmod{p};\]
hence by Lemma 2.3,
\[qa(x^m, x^n) = (m, n)a(x^{\rho^i}, x^{(q - 1)p^i}).\]

Since \(m\) and \(n\) are not divisible by \(\rho^i\), \((m, n) \equiv 0 \pmod{p}\) and (2.8) follows. If \(n > (q - 1)p^i\) then \(m < (q - 1)p^i\) and hence \(a(x^n, x^m) = 0\). By anticommutativity of \(\Delta\), (2.8) follows.

**Proof of (2.9).** Note that
\[(s \rho^i - \rho^i, \rho^i) = ((s - 1)\rho^i, \rho^i) \equiv (s - 1, 1) = s \not\equiv 0, \pmod{p}.\]
Therefore (2.9) is obtained on applying Lemma 2.3.

**Proof of the Main Lemma.** Let \(V\) be a normal monomial composed of primitive factors and of the same degree as \(z\). We consider two types of \(V:\)

(i) \(V\) has width greater than 1. Then we may write \(V = x^rS\), where \(x\) is the first factor of \(V\) and its multiplicity is \(r \geq 1\), and \(S \neq 1\). Put \(a(V) = a(x^r, S)\). Then using (2.3) and Lemma 2.2, we may write
\[\Delta''(a(V)V) = \sum_{V = (MN); M \neq 1, N \neq 1} a(V)[M, N]M \otimes N\]
\[= \sum_{V = (MN); M \neq 1, N \neq 1} a(M, N)M \otimes N.\]

It follows that if \(M \neq 1, N \neq 1\), and \(\nu(MN) = V\) then \(M \otimes N\) has zero coefficient in \(\Delta''(z - a(V)V)\).
(ii) \( V = x^{q^i} \), where \( q > 1 \) and \( q \equiv 0 \pmod{p} \). Put
\[
a(V) = a(x^{(q-1)^i}, x^{pi})/q.
\]
Using (2.3) we may write
\[
\Delta''(a(V)V) = \sum_{r,s>0; r+s=q} a(V)(r,s)x^{rpi} \otimes x^{spi}.
\]
Applying (2.9) to the terms for which \( s \neq 0 \) we may write
\[
\Delta''(a(V)V) = \sum_{s \neq 0} a(x^{rpi}, x^{spi})x^{rpi} \otimes x^{spi} \tag{2.10}
\]
\[+ \sum_{s=0} a(V)(r,s)x^{rpi} \otimes x^{spi}.
\]
We assert that if \( m+n = qp^i \) then \( x^m \otimes x^n \) has zero coefficient in \( \Delta''(z - a(V)V) \). In view of (2.8) only terms with \( m \) and \( n \) both divisible by \( p^i \) can occur. In view of (2.10) only terms with
\[
m = r^i, \quad n = s^i, \quad r + s = q, \quad r > 0, \quad s > 0, \quad s \equiv 0 \pmod{p}
\]
can occur. But \( s \equiv 0 \) and \( q \neq 0 \) imply \( r \neq 0 \). Thus, since \( \Delta \) is anticommutative,
\[
a(x^{rpi}, x^{spi}) = a(x^{spi}, x^{rpi}) = 0,
\]
and the assertion is proved.

Now define
\[
v = z - \sum a(V)V
\]
where the summation extends over all \( V \) of types (i) and (ii). Then \( v \) evidently has the properties asserted in the main lemma.

3. Proof of Theorem 1. Let \( H^*(T, Z_p) = Z_p[X] \); assume that the elements of \( X \cap \pi \) are primitive (see (2.2)). If \( H^*(T, Z_p) \) is not primitive then there is an element \( z \in X \) which is not in \( \pi \). Moreover, if we take \( z \) of lowest degree then \( \Delta''(z) \in \pi \otimes \pi \) and we may write (2.4). Since \( \Delta = \nabla^* \) is associative and anticommutative, there is an element \( v \in H \) with the properties specified by the main lemma. We shall show that \( v \) is primitive; this will produce a contradiction for it implies that \( z \in \pi \).

We shall make use of the following properties of \( St^i_p \):

\[
(3.1) \quad St^i_p: H^q(T, Z_p) \to H^{q+r(p-1)}(T, Z_p)
\]
where \( r = i \) if \( p = 2 \) and \( r = 2i \) if \( p \neq 2 \).

\[
(3.2) \quad St^i_p\Delta = \Delta St^i_p
\]
where

\[(3.3)\quad S_{p}(u \otimes w) = \sum_{i+j+k} S_{i}^{j}(u) \otimes S_{k}(w).\]

\[(3.4)\quad S_{i}^{j}(u) = \begin{cases} u, & \text{if } i = 0, \\ u^{r}, & \text{if } r = \text{degree of } u, \\ 0, & \text{if } r > \text{degree of } u, \end{cases}\]

where \(r\) is as defined above.

From (3.3) and (3.4) it follows that \(S_{i}^{j}\) commutes with \(\Delta'\) and hence also with \(\Delta''\) in view of (3.2). Thus

\[(3.5)\quad S_{i}^{j}A''(v) = S_{i}^{j}A''(v - z) + A''S_{i}^{j}(z).\]

Now consider the expression (2.5) for \(\Delta''(v)\). Let \(a(x^{m}, x^{n})x^{m} \otimes x^{n}\) be a summand such that \(md\) is maximum, where \(d\) is the degree of \(x\). Put

\[a = a(x^{m}, x^{n}), \quad m + n = p^{k}.\]

In (3.5) take \(i = mj\), where \(j = d\) if \(p = 2\), and \(2j = d\) if \(p \neq 2\). We shall prove:

A. \(S_{i}^{mj}(z) = 0\).

B. The coefficient of \(x^{mp} \otimes x^{n}\) in \(S_{i}^{mj}\Delta''(v)\) is \(a(x^{m}, x^{n})\).

C. The coefficient of \(x^{mp} \otimes x^{n}\) in \(S_{i}^{mj}\Delta''(v - z)\) is zero.

In view of (3.5) it follows from A, B, C that \(a(x^{m}, x^{n}) = 0\), and hence \(v\) is primitive.

**Proof of A.** The degrees of \(z\) and \(S_{i}^{mj}(z)\) are \(dp^{k}\) and \(d(p^{k} + m(p - 1))\), respectively. The latter is not a multiple of the former since \(p^{k} > m\), \(p - 1\). Thus A follows from the fact that \(S_{i}^{mj}(z)\) is in the subalgebra generated by \(z\).

**Proof of B.** We have

\[(3.6)\quad S_{i}^{mj}(ax^{m} \otimes x^{n}) = ax^{mp} \otimes x^{n} + \sum u_{i} \otimes w_{i},\]

where the degrees of the \(u_{i}\) are less than \(mdp\). It remains to show that no other summand \(by^{r} \otimes y^{s}\) in \(\Delta''(v)\) can contribute to the coefficient of \(x^{mp} \otimes x^{n}\). If the degree of \(y^{r}\) is less than \(md\) this is clear; if the degree of \(y^{r}\) is \(md\) then, writing a similar expression to (3.6) for \(S_{i}^{mj}(by^{r} \otimes y^{s})\), we see that only \(by^{r} \otimes y^{s}\) has the same bidegree as \(x^{mp} \otimes x^{n}\). But if \(y^{r} \neq y^{s} \neq x^{m} \otimes x^{n}\) then \(y \neq x\) or \(r \neq m\), and hence \(S_{i}^{mj}(by^{r} \otimes y^{s})\) contributes nothing to the coefficient of \(x^{mp} \otimes x^{n}\).

**Proof of C.** Combining (2.4) and (2.5) we may write
(3.7) \[ \Delta''(v - z) = - \sum a(M, N) M \otimes N; \]

note that \( M \otimes N \) has the property that \( MN \neq \gamma \) for \( \gamma \in X \). Consider such a term \( M \otimes N \). Let \((d_1, d_2)\) be its bidegree, and \( c_{M,N} \) the coefficient of \( x^{mp} \otimes x^n \) in \( S^{p_{\mu}}(M \otimes N) \). If \( d_1 < md \) then it is clear that \( c_{M,N} = 0 \).

If \( d_1 = md \) then the only term in \( S^{p_{\mu}}(M \otimes N) \) with the same bidegree as \( x^{mp} \otimes x^n \) is \( M^p \otimes N \). In view of the restriction on \( MN \), \( M^p \otimes N \neq x^{mp} \otimes x^n \), and hence \( c_{M,N} = 0 \). Finally, we complete the proof of \( C \) and hence of Theorem 1 by showing that if \( d_1 > md \) then \( a(M, N) = 0 \).

Let \( M \otimes N \) be such that \( d_1 \) is maximum. In (3.5) take \( i = d_1 \) if \( p = 2 \), and \( i = d_1/2 \) if \( p \neq 2 \) (the latter is possible since if \( p \neq 2 \), \( M \) has even degree by (2.1)). We assert:

A'. \( S^{d_1}_p(z) = 0 \).

B'. The coefficient of \( M^p \otimes N \) in \( S^{d_1}_p \Delta''(v - z) \) is \(-a(M, N)\).

C'. The coefficient of \( M^p \otimes N \) in \( S^{d_1}_p \Delta''(v) \) is zero.

In view of (3.5), A', B', C' imply \( a(M, N) = 0 \). The proof of C' follows immediately from \( md < d_1 \). For if \((e_1, e_2)\) is the bidegree of a term in \( S^{d_1}_p \Delta''(v) \) then \( e_1 \) is at most \( md + d_1(p - 1) < pd_1 \). The proof of B' is very similar to the proof of B and we omit the details. To prove A' it suffices to show that the degree of \( S^{d_1}_p(z) \) which is \( dp^k + d_1(p - 1) \) is not a multiple of \( dp^k \) (the degree of \( z \)) or, equivalently, that \( d_1(p - 1) \) is not a multiple of \( dp^k \).

Consider \( a(x^m, x^n) \) again and put \( m = qp^i \), where \( q \neq 0 \pmod{p} \). By Lemma 2.1, we have
\[ qa(x^m, x^n) = (p^k - p^i - n, n)a(x^{p^i}, x^{p^k-p^i}). \]

Thus if \( a(x^m, x^n) \neq 0 \) then \( a(x^{p^i}, x^{p^k-p^i}) \neq 0 \). By anticommutativity of \( \Delta \), then \( a(x^{p^k-p^i}, x^{p^i}) \neq 0 \). Since the term \( x^m \otimes x^n \) was chosen so that \( md \) was maximum, it follows that
\[ md \geq (p^k - p^i)d \geq (p^k - p^{k-1})d. \]

Combining this with the inequalities
\[ dp^k > d_1 > md \]

and multiplying through by \( (p - 1)/dp^k \) gives
\[ (p - 1) > \frac{d_1(p - 1)}{dp^k} > \left(1 - \frac{1}{p^k}\right)(p - 1). \]

Thus \( d_1(p - 1)/dp^k \) is not an integer.

**Bibliography**


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SOME GLOBAL PROPERTIES OF HYPERSURFACES

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1. Introduction. The translation theorem of Hopf [1] has been extended by Hsiung [2] and Voss [4] independently to hypersurfaces and by Hsü [3] to other elementary transformations. The purpose of this paper is to extend to hypersurfaces in \((n+1)\)-dimensional Euclidean space some results obtained by Hsü [3] for the case \(n = 2\).

All hypersurfaces mentioned will be assumed to be twice differentiably imbedded in an \((n+1)\)-dimensional Euclidean space \(E^{n+1}(n+1 \geq 3)\). The notation used will be that of Hsiung [2]. In particular, \(X, N, M_1, A\) denote the position vector, unit inner normal, first mean curvature, and area for the hypersurface \(V^n\). Corresponding quantities for other hypersurfaces will be denoted by \(*\), or by primes.

Considerable use will be made of the vector product defined by Hsiung [2]. Namely, if \(i_1, \ldots, i_{n+1}\) denotes a fixed frame of mutually orthogonal unit vectors and \(A_1, \ldots, A_n\) are \(n\) vectors whose components in this frame are \(A_\alpha^i (i=1, \ldots, n; \alpha = 1, \ldots, n+1)\), the vector product is defined by

\[
A_1 \times \cdots \times A_n = (-1)^n \begin{vmatrix}
  i_1 & i_2 & \cdots & i_{n+1} \\
  A_1^1 & A_1^2 & \cdots & A_1^{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  A_n^1 & A_n^2 & \cdots & A_n^{n+1}
\end{vmatrix}.
\]

1 This is a portion of a master's thesis at the University of Oklahoma directed by Professor T. K. Pan.