of z and each S_j is even. Let s be the number of points of the orbit and q any index in Q. For $r<s$, $(hz)^r(p, q)$ differs from (p, q) in the first coordinate; but $(hz)^s(p, q)=(p, q)$. Thus every element of G has an odd cycle. As we noted above, this implies [1] the existence of a fair game of $2^k(2^l-1)$ players.

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ON INDUCED TOPOLOGIES IN QUASI-REFLEXIVE BANACH SPACES[1]

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1. **Introduction.** Let π denote the canonical isomorphism of a Banach space X into its second conjugate space X^{**}. An example is given by James [4] of a space X for which X is separable, X is not reflexive, X is isomorphic to X^{**}, and $X^{**}/\pi(X)$ is one-dimensional. Civin and Yood undertook a more complete investigation of Banach spaces X such that $X^{**}/\pi(X)$ is (finite) n-dimensional and called such spaces quasi-reflexive Banach spaces of order n. If Q is a subset of X^*, let $\sigma(X, Q)$ denote the least fine topology for X such that all $x^*\in Q$ are continuous. In [1] Civin and Yood establish the following result.

Theorem A. *The following statements are equivalent:*

(1) *X is quasi-reflexive of order n.*

(2) *There is an equivalent norm for X such that $X^*=Q\oplus R$ where Q is a total closed linear manifold such that the unit ball of X is compact in $\sigma(X, Q)$ and R is an n-dimensional linear manifold.*

It is the purpose of this paper to study properties of the topologies $\sigma(X, Q)$, where $X^*=Q\oplus R$, Q is a total closed linear manifold, and

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R is n-dimensional. It is shown that $\sigma(X, Q)$ is nothing more than the w^*-topology on X when X is considered as the conjugate space of Q.

2. **Notation.** Let X be a Banach space. Let π be the canonical isomorphism of X into X^{**}, its second conjugate space. For a subset A of X, A^+ will designate the annihilator of A in X^*, and A^{++} the annihilator of A^+ in X^{**}. For a set B in X^*, B^- will denote the annihilator of B in X. When we write $X = C \oplus D$, we shall mean that C and D are closed linear manifolds of X, that X is the linear span of C and D, and $C \cap D = 0$. We define $S_r = \{x \in X : \|x\| \leqq r\}$.

3. **Preliminary results.** If X is a quasi-reflexive Banach space of order n, then $X^{**} = \pi(X) \oplus L$ where L is an n-dimensional linear manifold. Civin and Yood note that $X^* = Q \oplus R$ where $Q = L^-$ is total and R is n-dimensional. In the proof of Theorem A, they show that for $Q = L^-$ there is an equivalent norm for X such that the unit ball of X is compact in $\sigma(X, Q)$.

The following question can then be posed. If X is a quasi-reflexive space of order n and $X^* = Q_0 \oplus R_0$ where Q_0 is total and R_0 is n-dimensional, is there an equivalent norm for X in which the unit ball is compact in $\sigma(X, Q_0)$? The following theorem shows that all decompositions of X^* of the above type arise from considering the annihilators of the n-dimensional pieces of the second conjugate space of X.

3.1. THEOREM. *If X is a quasi-reflexive Banach space of order n and if $X^* = Q_0 \oplus S_0$ where Q_0 is total and S_0 is n-dimensional, then*:

(i) $X^{**} = \pi(X) \oplus Q_0^+$,

(ii) *there is an equivalent norm for X such that the unit ball of X is compact in $\sigma(X, Q_0)$*,

(iii) $\|x\| = \sup_{x^* \in Q_0; \|x^*\| = 1} |x^*(x)|$ *if X has the norm for which the unit ball is compact in $\sigma(X, Q)$.*

PROOF. (i) Suppose that $x^{**} \in \pi(X) \cap Q_0^+$. Then $x^{**} = \pi(x)$ for some $x \in X$ and for all $y^* \in Q_0$, $x^{**}(y^*) = y^*(x) = 0$. Since Q_0 is total, $x = 0$, and hence $\pi(X) \cap Q_0^+ = 0$. Since $X^{**}/\pi(X)$ has dimension n, it follows that Q_0^+ has dimension $r \leqq n$. Let $x_1^{**}, x_2^{**}, \cdots, x_r^{**}$ be a basis for Q_0^+ and select $x_1^*, x_2^*, \cdots, x_r^* \in X^*$ such that $x_i^{**}(x_j^*) = \delta_{ij}$, $i, j = 1, 2, \cdots, r$. Let R be the subspace of X^* generated by $x_1^*, \cdots, x_r^*$. It is easily seen that $X^* = Q_0 \oplus R$ and thus X^*/Q_0 has dimension r. But $X^* = Q_0 \oplus S_0$ where S_0 is n-dimensional, so X^*/Q_0 has dimension n. Hence $r = n$.

(ii) Since $X^{**}=\pi(X)\oplus Q_0^+$, the result follows immediately from the proof of Theorem A.

(iii) This follows immediately from Theorem 7 of [**3**].

In view of 3.1, we adopt the following convention. When we say that X is a quasi-reflexive Banach space, $X^*=Q\oplus R$ where Q is total and R is n dimensional, we shall always mean that X is to be considered in its equivalent norm so that its unit ball is compact in $\sigma(X, Q)$.

4. Induced topologies. In this section the topologies induced on X by the decompositions $X^*=Q\oplus R$, Q total, R n-dimensional, are characterized as w^*-topologies.

4.1. THEOREM. *If X is a quasi-reflexive Banach space, $X^*=Q\oplus R$ where Q is total and R is n-dimensional, then X is equivalent to Q^* under the mapping $\nu: X\rightarrow Q^*$ defined by $\nu(x)(x^*)=x^*(x)$, all $x^*\in Q$.*

PROOF. $\nu(x)$ is the contraction of $\pi(x)$ to Q. This is linear and 1-1, since Q is total. By Theorem 9 of [**2**], $\pi(x)$ and its contraction to Q have the same norm.

Hence $\sigma(X, Q)$ is merely the w^*-topology on X when X is considered as the conjugate space of Q and properties which hold for general conjugate spaces thus hold for quasi-reflexive Banach spaces.

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