ON PARTIALLY ORDERED SETS POSSESSING A UNIQUE ORDER-COMPATIBLE TOPOLOGY

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1. **Introduction.** Let X be a partially ordered set (poset) with respect to a relation $\leqq$, and possessing least and greatest elements O and I respectively. Let us call a subset S of X *up-directed* (*down-directed*) if and only if for all $x \in S$ and $y \in S$ there exists $z \in S$ such that $z \geqq x$, $z \geqq y$ ($z \leqq x$, $z \leqq y$). Following McShane [2], we call a subset K of X *Dedekind-closed* if and only if whenever S is an up-directed subset of K and $y = \text{l.u.b. } (S)$, or S is a down-directed subset of K and $y = \text{g.l.b. } (S)$, we have $y \in K$. Let $\mathfrak{D}$ denote the topology on X whose closed sets are the Dedekind-closed subsets of X. Let $\mathfrak{J}$ denote the well-known interval topology on X, which is obtained by taking all sets of the form $[a, b] = \{x \in X \mid a \leqq x \leqq b\}$ as a sub-base for the closed sets. Continuing an investigation which was begun in [5], we shall call a topology $\mathfrak{I}$ on X *order-compatible* if and only if $\mathfrak{J} \leqq \mathfrak{I} \leqq \mathfrak{D}$. X is said to have a unique order-compatible topology if and only if its $\mathfrak{J}$ and $\mathfrak{D}$ topologies are identical. In [5] we obtained a simple sufficient condition for a poset to possess a unique order-compatible topology. This result has recently been strengthened by Naito [3]. Let us call a subset K of X *diverse* if and only if $x \in K$, $y \in K$, and $x \neq y$ imply $x \not< y$. Naito has shown that *if a poset X contains no infinite diverse subset, then it possesses a unique order-compatible topology.*

The purpose of the present note is to obtain a condition both necessary and sufficient for a poset to have a unique order-compatible topology. However, such a condition has been obtained only for posets satisfying a certain countability restriction (the question of finding a condition which holds in general still remains open). We show that a poset X satisfying such a hypothesis has a unique order-compatible topology if and only if every diverse subset of X is $\mathfrak{J}$-closed. With a slight strengthening of our hypothesis we obtain a sufficient condition for this unique topology to be metrizable. We conclude with an investigation of some properties of the order structure of a poset in which every diverse subset is $\mathfrak{J}$-closed.

2. **Main results.** We first state without proof a lemma which was obtained by Naito [3] and which is an improvement of Lemma 6 of

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[5]. Notation and terminology are the same as in [5]. The obvious dual formulation may be left to the reader.

LEMMA 1 (NAITO). *Let X be a poset containing no infinite diverse subset, and let $(f(\alpha), \alpha \in A)$ be a net with range $(f) = S \subset X$. Let y be an element of X such that y is the l.u.b. of the range of every subnet of f. Then there exists an up-directed set $M \subset S$ such that $y = $ l.u.b. (M).*

We now prove our main results.

THEOREM 1. *Let X be a poset such that the space $(X, \mathfrak{g})$ satisfies the first axiom of countability. Then X has a unique order-compatible topology if and only if every diverse subset of X is $\mathfrak{g}$-closed. Furthermore, X is a Hausdorff space with respect to this topology.*

PROOF. A diverse subset of any poset X is always $\mathfrak{D}$-closed. Hence, if $\mathfrak{g} = \mathfrak{D}$, every diverse subset of X is $\mathfrak{g}$-closed.

To prove the converse, suppose that every diverse subset of X is $\mathfrak{g}$-closed, and let K be a $\mathfrak{D}$-closed subset. Let $\{x_n\}$ be a sequence of elements of K which converges to an element $y \in X$ in the interval topology. We may assume that $x_n \neq y$ for all n. We must show that $y \in K$.

We first show that there exists a subsequence of $\{x_n\}$ each member of which is comparable with y. For if this is not the case, then x_n is incomparable with y for all sufficiently large n, and by Lemma 4 of [5] there exists an infinite diverse subset of X which is contained in the range of $\{x_n\}$. But this means that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ whose range R is diverse. Since by hypothesis R is $\mathfrak{g}$-closed, and $y \notin R$, we have $\lim x_{n_k} \neq y$ in the $\mathfrak{g}$ topology, which is a contradiction.

Let us assume that each member of the sequence $\{x_n\}$ itself is comparable with y. We may assume, furthermore, that $x_n < y$ for all n (for if this is not eventually true, then there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} > y$ for all k, and the obvious dual proof will apply). Let S be the range of $\{x_n\}$. By the argument of the previous paragraph, S contains no infinite diverse subset. Furthermore, by Lemmas 3 and 5 of [5], y is the l.u.b. of the range of every subnet of $\{x_n\}$. Hence Lemma 1 applies, and we conclude that S contains an up-directed subset M with $y = $ l.u.b. (M). Since $M \subset K$ and K is $\mathfrak{D}$-closed, we have $y \in K$. Hence K is $\mathfrak{g}$-closed, and $\mathfrak{g} = \mathfrak{D}$.

It remains to show, under the above hypotheses, that the topology $\mathfrak{g}$ is Hausdorff. Let $\{z_n\}$ be any sequence in X which converges in the $\mathfrak{g}$ topology to an element $y \in X$. By the arguments of the previous paragraphs we may assume that $z_n < y$ for all n; and again by Lemmas

3 and 5 of [5] it follows that $y = \text{l.u.b.}\ (z_n)$. Hence any $\mathcal{J}$-convergent sequence of elements of X has a unique limit. Since $(X, \mathcal{J})$ satisfies the first axiom of countability, this implies that $(X, \mathcal{J})$ is a Hausdorff space.

If F is a closed subset of a topological space $(X, \mathfrak{J})$, we shall say that F has a *countable system of neighborhoods* if and only if there exists a countable family $\mathfrak{U} \subset \mathfrak{J}$ such that whenever $F \subset T$ and $T \in \mathfrak{J}$, there is a $U \in \mathfrak{U}$ with $F \subset U \subset T$. (Clearly this condition implies the first axiom of countability.)

We now have the following theorem.

THEOREM 2. *Let X be a poset in which every closed interval $[a, b]$ has a countable system of neighborhoods in the topology $\mathcal{J}$. If X has a unique order-compatible topology, then X is a regular space with respect to this topology.*

PROOF. It is sufficient to show that if B is a member of a base $\mathfrak{B}$ for the closed sets of the topology $\mathcal{J}$, and $c \notin B$, then there exist disjoint $\mathcal{J}$-open sets U and V such that $B \subset U$ and $c \in V$. (For if F is an arbitrary $\mathcal{J}$-closed subset of X and $c \notin F$, then there exists $B \in \mathfrak{B}$ with $F \subset B$ and $c \notin B$.) Since the collection of all finite unions of closed intervals of X is a base for the closed sets of the topology $\mathcal{J}$, it follows that it is sufficient to show that whenever $[a, b]$ is a closed interval, and $c \notin [a, b]$, then there exist disjoint $\mathcal{J}$-open sets U and V with $[a, b] \subset U$ and $c \in V$. Suppose that this is not true for some interval $[a, b]$ in X and some $c \notin [a, b]$. Let $\{U_n | n = 1, 2, \cdots\}$ and $\{V_n | n = 1, 2, \cdots\}$ be decreasing sequences of open sets which form countable neighborhood systems for $[a, b]$ and c, respectively. Then $U_n \cap V_m$ is nonvoid for all m and n. For each $n = 1, 2, \cdots$, choose $x_n \in U_n \cap V_n$. Then the sequence $\{x_n\}$ converges to c in the $\mathcal{J}$ topology and is also eventually in every $\mathcal{J}$-open set which contains $[a, b]$. As in the proof of Theorem 1, x_n must be comparable with c for all sufficiently large n We may therefore again assume that $x_n < c$ for all n. It then follows, again using Lemmas 3 and 5 of [5], that $c = \text{l.u.b.}\ (x_n)$. Furthermore, since $[a, b]$ is closed and $c \notin [a, b]$, there exists n_0 such that $x_n \notin [a, b]$ for all $n \geqq n_0$. Since c is the unique limit of the sequence $\{x_n\}$, the set $F = \{x_n | n \geqq n_0\} \cup \{c\}$ is $\mathcal{J}$-closed, and also disjoint from $[a, b]$. Let G = complement of F. Then G is an $\mathcal{J}$-open set containing $[a, b]$ but $\{x_n\}$ is not eventually in G, a contradiction.

The question remains open as to whether Theorems 1 and 2 remain valid without some countability assumptions.

It is natural to ask for a purely "order-theoretic" property of X which is sufficient to imply the topological countability hypotheses of

Theorems 1 and 2. A convenient property to consider is the following.

PROPERTY C. There exists a countable subset R of X such that, whenever J and K are disjoint closed intervals in X, there exist $r \in R$ and $s \in R$ with J disjoint from $[r, s]$ and $K \subset [r, s]$.

We may now prove

LEMMA 2. *If a poset X has property* C, *then* (i) $(X, \mathcal{I})$ *satisfies the second axiom of countability, and* (ii) *every closed interval* $[a, b]$ *in X has a countable system of neighborhoods in the topology* $\mathcal{I}$.

PROOF. (i) follows immediately from the observation that the family of all finite unions of closed intervals of the form $[r, s]$, for $r \in R$ and $s \in R$, is a base for the closed sets of $(X, \mathcal{I})$. To prove (ii), let $\mathcal{B}$ be the family of all finite unions of sets $[r, s]$ such that $r \in R$, $s \in R$, and $[r, s]$ is disjoint from $[a, b]$. Then the family of all complements of members of $\mathcal{B}$ is a countable neighborhood system for $[a, b]$ in the topology $\mathcal{I}$.

Lemma 2, Theorems 1 and 2, and the well-known metrization theorem of Urysohn now imply

THEOREM 3. *Let X be a poset with property* C *in which every diverse subset is $\mathcal{I}$-closed. Then X has a unique order-compatible topology which is metrizable.*

3. Diverse subsets which are closed in the interval topology.

It may be of interest also to obtain an "order-theoretic" property which characterizes a poset in which all diverse subsets are $\mathcal{I}$-closed. To do this, we shall first give a characterization of the $\mathcal{I}$-convergent nets in X. Our terminology in regard to nets is that of [1]. In particular, we follow [1] in our use of the terms "eventually" and "frequently."

The following notation will be convenient. If $K \subset X$, we shall write $K^+ = \{x \in X \mid x \leqq y \text{ for all } y \in K\}$ and $K^* = \{x \in X \mid x \geqq y \text{ for all } y \in K\}$. If f is a net in X, let Γ_f denote the set of all cofinal subnets of f. Then we define

$$M_f = \bigcup \{[\text{range}(g)]^+ \mid g \in \Gamma_f\},$$

$$N_f = \bigcup \{[\text{range}(g)]^* \mid g \in \Gamma_f\}.$$

THEOREM 4. *A net f in the poset X converges to an element y in the topology $\mathcal{I}$ if and only if* $y \in M_f^* \cap N_f^+$.

PROOF. Suppose that f $\mathcal{I}$-converges to y and that $y \notin M_f^* \cap N_f^+$. Then there exists $m \in M_f$ with $y \ngeqq m$, or there exists $n \in N_f$ with $y \nleqq n$. In either case there exists a closed interval J in X such that J

contains the range of a cofinal subnet of f and $y \notin J$. Then $X-J$ (the complement of J with respect to X) is an $\mathcal{J}$-open neighborhood of y, and by hypothesis f is eventually in $X-J$. But this means that f is not frequently in J, a contradiction.

To prove the converse, suppose that f does not converge to y in the $\mathcal{J}$ topology. Then there exists an $\mathcal{J}$-open set U such that $y \in U$ and f is frequently in $X-U$. But $X-U$ is the intersection of a family of members of the usual closed base for the topology $\mathcal{J}$. Hence there exists a member B of this closed base such that $y \notin B$ and f is frequently in B. But B is of the form $\bigcup\{J_i \mid i=1, 2, \cdots, n\}$, where each J_i is a closed interval; and f is frequently in B implies that f is frequently in some J_k $(k=1, 2, \cdots, n)$. If $J_k=[a_k, b_k]$, then $a_k \in M_f$, $b_k \in N_f$. But $y \notin J_k$, and hence $y \ngeqq a_k$ or $y \nleqq b_k$. In either case we have $y \notin M_f^* \cap N_f^+$.

Now let L be any infinite diverse subset of X. Let $\mathfrak{F}(L)$ denote the family of all sets of the form $L-F$, where F is a finite subset of L. Let us define

$$A_L = \bigcup \{K^+ \mid K \in \mathfrak{F}(L)\}, \qquad B_L = \bigcup \{K^* \mid K \in \mathfrak{F}(L)\}.$$

We then have the following theorem.

THEOREM 5. *An infinite diverse subset L of a poset X is $\mathcal{J}$-closed if and only if $A_L^* \cap B_L^+ \subset L$.*

PROOF. Suppose that L is $\mathcal{J}$-closed. Let $D=\{(x, K) \mid K \in \mathfrak{F}(L)$ and $x \in K\}$. The set D may be up-directed by defining $(x_1, K_1) \leqq (x_2, K_2)$ if and only if $K_1 \supset K_2$. If we define $f(x, K)=x$, then f is a net on D with values in L. We shall show that for this net, f, $M_f^* \cap N_f^+ = A_L^* \cap B_L^+$. Since $\{f(x, K) \mid (x, K) \geqq (x_0, K_0)\} = K_0$, it follows that each $K \in \mathfrak{F}(L)$ is the range of a cofinal subnet of f (actually a residual subnet). Hence $A_L \subset M_f$, $B_L \subset N_f$. But then $A_L^* \supset M_f^*$, $B_L^+ \supset N_f^+$, and $A_L^* \cap B_L^+ \supset M_f^* \cap N_f^+$. To prove the reverse inclusion, note that the range of any cofinal subnet g of f obviously contains some $K_0 \in \mathfrak{F}(L)$. Hence $[\text{range }(g)]^+ \subset K_0^+$, and dually. Then $M_f \subset A_L$, $N_f \subset B_L$, and $A_L^* \cap B_L^+ \subset M_f^* \cap N_f^+$. Hence $A_L^* \cap B_L^+ = M_f^* \cap N_f^+$, and by Theorem 4 we must have $A_L^* \cap B_L^+ \subset L$.

To prove the converse it is convenient to use the terminology of filters. If $\mathfrak{U}$ is a filter on L, let us define $P(\mathfrak{U}) = \bigcup\{S^+ \mid S \in \mathfrak{U}\}$, $Q(\mathfrak{U}) = \bigcup\{S^* \mid S \in \mathfrak{U}\}$. Following Ward [4], we say that an element x of X is *medial* for $\mathfrak{U}$ if and only if $x \in [P(\mathfrak{U})]^* \cap [Q(\mathfrak{U})]^+$. Ward [4] has shown that *x is medial for an ultrafilter $\mathfrak{U}$ on X if and only if $\mathfrak{U}$ is $\mathcal{J}$-convergent to x.*

Now to show that L is $\mathcal{J}$-closed it is sufficient to show that if $\mathfrak{U}$ is any ultrafilter on L, then $\mathfrak{U}$ does not converge to any point of $X-L$. We distinguish two cases. First, suppose that there exists $y\in L$ with $y\in S$ for all $S\in\mathfrak{U}$. Then we must have $\{y\}\in\mathfrak{U}$, and $\mathfrak{U}$ cannot $\mathfrak{J}$-converge to any point of X other than y (since $(X, \mathfrak{J})$ is a T_1-space). Suppose then that there exists no $y\in L$ with $y\in S$ for all $S\in\mathfrak{U}$. Then $\mathfrak{U}$ must contain the filter $\mathfrak{F}=\mathfrak{F}(L)$. It is easy to verify that $\mathfrak{U}\supset\mathfrak{F}$ implies that $[P(\mathfrak{U})]^*\cap[Q(\mathfrak{U})]^+\subset[P(\mathfrak{F})]^*\cap[Q(\mathfrak{F})]^+=A_L^*\cap B_L^+$. But by hypothesis $A_L^*\cap B_L^+\subset L$. Thus $x\in L$ whenever x is medial for $\mathfrak{U}$, which completes the proof.

THEOREM 6. *Every diverse subset of a poset X is $\mathfrak{J}$-closed if and only if $A_L^*\cap B_L^+$ is empty for every infinite diverse subset L.*

PROOF. Suppose that every diverse subset of X is $\mathfrak{J}$-closed, and that for some infinite diverse L there is an element $z\in A_L^*\cap B_L^+$. By Theorem 5, $z\in L$. Let $H=L-\{z\}$. Since $\mathfrak{F}(H)\subset\mathfrak{F}(L)$, we have $A_H\subset A_L$, $B_H\subset B_L$. Hence $A_H^*\supset A_L^*$, $B_H^+\supset B_L^+$, and $z\in A_H^*\cap B_H^+$. But H is $\mathfrak{J}$-closed and $z\notin H$, contradicting Theorem 5. The converse proposition follows trivially from Theorem 5.

Theorem 6 suggests that it is slightly "pathological" for a poset to contain infinite diverse subsets all of which are $\mathfrak{J}$-closed. However, the following is a simple example of such a poset. Consider the following sets of points in the complex plane. Let $X_1=\{(1/n, 0)\mid n=1, 2, \cdots\}$, $X_2=\{(0, 1/n)\mid n=\pm 1, \pm 2, \cdots\}$. Let $X=X_1\cup X_2$. We partially order X as follows. Define $(0, 1/m)\leqq(0, 1/n)$ if and only if $1/m\leqq 1/n$ in the usual ordering. For $n>0$, define $(1/m, 0)<(0, 1/n)$ if and only if $1/m\leqq 1/n$. For $n<0$, define $(1/m, 0)>(0, 1/n)$ if and only if $1/m\leqq 1/n$. The set X_1 is diverse. With this ordering it is clear that X satisfies the hypotheses of Theorem 3.

REFERENCES

1. J. L. Kelley, *General topology*, Van Nostrand, 1955.

2. E. J. McShane, *Order-preserving maps and integration processes*, Annals of Mathematics Studies, vol. 31, Princeton, 1953.

3. T. Naito, *On a problem of Wolk in interval topologies*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 156–158.

4. A. J. Ward, *On relations between certain intrinsic topologies in partially ordered sets*, Proc. Cambridge Philos. Soc. vol. 51 (1955) pp. 254–261.

5. E. S. Wolk, *Order-compatible topologies on a partially ordered set*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 524–529.

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