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A CHARACTERIZATION OF ALGEBRAIC NUMBER FIELDS WITH CLASS NUMBER TWO[1]

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Let $Z=R(\theta)$ denote an algebraic number field over the rationals with class number h. It is familiar that $h=1$ if and only if unique factorization into prime holds for the integers of Z. For fields with $h\leqq 2$ we have the following criterion.

Theorem. *The algebraic number field Z has class number $\leqq 2$ if and only if for every nonzero integer $\alpha\in Z$ the number of primes π_j in every factorization*

$$\alpha = \pi_1\pi_2 \cdots \pi_k \tag{1}$$

depends only on α.

Suppose first that $h=2$ and consider the factorization into prime ideals

$$(\alpha) = \mathfrak{p}_1 \cdots \mathfrak{p}_s\mathfrak{r}_1 \cdots \mathfrak{r}_t, \tag{2}$$

where the $\mathfrak{p}_j$ are principal ideals while the $\mathfrak{r}_j$ are not. Then

$$\mathfrak{p}_j = (\pi_j) \qquad (j = 1, \cdots, s).$$

Since $h=2$, it follows that

$$\mathfrak{r}_i\mathfrak{r}_j = (\rho_{ij}) \qquad (i, j = 1, \cdots, t);$$

moreover t must be even, $=2u$, say. Thus every factorization into primes implied by (2), for example

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$$\alpha = \epsilon\pi_1 \cdots \pi_s\rho_{12} \cdots \rho_{t-1,t},$$

where ϵ is a unit, will contain exactly $s+u$ primes.

We now show that when $h>2$, there occur factorizations (1) with different values of k. The proof makes use of the fact that every class of ideals contains at least one prime ideal. (For proof of a much stronger result see [1]).

Assume first the existence of a class A of period $m>2$. Let $\mathfrak{p}$ be a prime ideal in A and $\mathfrak{p}'$ a prime ideal in A^{-1}. Then we have

$$\mathfrak{p}^m = (\pi), \qquad \mathfrak{p}'^m = (\pi)', \qquad \mathfrak{p}\mathfrak{p}' = (\pi_1), \tag{3}$$

and it is easily verified that π, π', π_1 are primes. Clearly (3) implies

$$\pi_1^m = \epsilon\pi\pi', \tag{4}$$

where ϵ is a unit.

In the next place assume the existence of two classes A_1, A_2 each of period 2 such that $A_3=A_1A_2$ is not principal. Choose prime ideals $\mathfrak{p}_j \in A_j$ $(j=1, 2, 3)$. Then we have

$$\mathfrak{p}_j^2 = (\pi_j) \ (j = 1, 2, 3), \qquad \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 = (\pi), \tag{5}$$

and again it is easily verified that π_1, π_2, π_3, π are all primes. From (5) we get

$$\pi^2 = \pi_1\pi_2\pi_3. \tag{6}$$

Using (5) and (6) it is evident that when $h>2$, the number of primes k in (1) is not independent of the factorization.

Since the case $h=1$ requires no further discussion, this completes the proof of the theorem.

Reference

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