INSERTION OF $\pm$ SIGNS IN e^x

DONALD J. NEWMAN

We consider the series $e^x = \sum x^n/\lfloor n$ and investigate to what extent its "magnitude" can be lessened by a propitious choice of ± 1 coefficients.

This investigation began with the following elementary question.

I. Can ϵ_n be chosen ($\epsilon_n = \pm 1$) such that

$$\sum \frac{\epsilon_n x^n}{\lfloor n} \to 0 \text{ as } x \to -\infty \quad \textit{and} \quad \text{as } x \to +\infty?$$

We now prove a rather general theorem which contains the negative answer to I.

THEOREM.

$$f(x) = \sum \frac{\epsilon_n x^n}{\lfloor n} = Oe^{\rho x}, \qquad \rho < 1, \textit{ as } x \to +\infty$$

if and only if, for sufficiently large n, the ϵ_n are periodic, with even period $2K$, and $\epsilon_{n+1}+\epsilon_{n+2}+\cdots+\epsilon_{n+2K}=0$.

PROOF. The "if" part is, of course, the easy half, for if the ϵ_n are periodic of period $2K$ then $f(x)$ is a linear combination of the $e^{w^r x}$ where $w = e^{\pi i/K}$, $0 \leq r < 2K$. If this combination is

$$f(x) = \sum_{r=0}^{2K-1} a_r e^{w^r x}$$

then

$$\epsilon_n = \sum_{r=0}^{2K-1} a_r w^{rn} \quad \text{and so} \quad \epsilon_{n+1} + \cdots \epsilon_{n+2K} = 2Ka_0$$

hence $a_0 = 0$ and so $|f(x)| \leq Me^{(\cos \pi/k)x}$. We now prove the "only if" statement:

Consider the expression

$$\frac{1}{Z}\int_0^\infty f(x)e^{-x/Z}dx = g(Z).$$

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First note that, since $|f(x)| \leqq Me^{\rho x}$, $g(Z)$ is analytic for Re $(1/Z) > \rho$. Next note that, for Re $(1/Z) > 1$,

$$g(Z) = \frac{1}{Z}\int_0^\infty \sum \epsilon_n \frac{x^n}{\underline{|n}} e^{-x/Z}dx = \frac{1}{Z}\sum \epsilon_n \int_0^\infty \frac{x^n}{\underline{|n}} e^{-x/Z}dx,$$

the inversion being justified by the bounded convergence theorem since

$$\int_0^\infty \sum \left|\epsilon_n \frac{x^n}{\underline{|n}}\right| \left| e^{-x/Z} \right| dx = \int_0^\infty e^x e^{-x\mathrm{Re}\, 1/Z}dx < \infty.$$

Finally, then, for

$$\mathrm{Re}\frac{1}{Z} > 1, \qquad g(Z) = \sum \frac{\epsilon_n}{Z}\int_0^\infty \frac{x^n}{\underline{|n}} e^{-x/Z}dx = \sum \epsilon_n Z^n.$$

The conclusion is that $\sum \epsilon_n Z^n$ can be continued past the unit circle [into Re $(1/Z) > \rho$, in fact.]

It is a theorem of Carlson [1], however, that if the a_n are integers, $\sum a_n Z^n$ has radius of convergence $=1$, and $\sum a_n Z^n$ is not a rational function then $|Z| = 1$ is the natural boundary for $\sum a_n Z^n$.

The conclusion for us, then, is that $\sum \epsilon_n Z^n$ is a rational function! It therefore follows that, for n past a certain point, the ϵ_n satisfy a finite linear recurrence relation. Because of this and the fact that the ϵ_n take on only a finite number of values ($\epsilon_n = \pm 1$), it follows that the ϵ_n are periodic past a certain point. Hence

$$g(Z) = \sum \epsilon_n Z^n = P(Z) + \frac{\delta_0 + \delta_1 Z + \cdots \delta_{M-1}Z^{M-1}}{1 - Z^M}$$

where P is a polynomial, $\delta_j = \pm 1$, and M the period of ϵ_n. For sufficiently large n we obtain

$$\epsilon_{n+1} + \epsilon_{n+2} + \cdots \epsilon_{n+M} = \delta_0 + \delta_1 + \cdots \delta_{M-1}$$

but

$$\delta_0 + \delta_1 + \cdots \delta_{M-1} = \lim_{Z\to 1} (1 - Z^M)g(Z)$$

and the latter is 0 since $g(Z)$ is regular at 1. Hence $\epsilon_{n+1} + \cdots \epsilon_{n+M} = 0$, and in particular $M = 2K$, K integral and the proof is complete.

The negative answer to question I is now easily given. If $f(x) \to 0$ at $\pm\infty$ then, since by our theorem, $f(x)$ is a trigonometric polynomial, we have $f(x) = Oe^{-\delta|x|}$, $x \to \pm\infty$. Since, however, $|f(x)| \leqq e^{|x|}$ for all complex x, a Phragmen-Lindelof theorem [2] gives the final contradiction.

Certain other corollaries can be reaped. It can be shown, e.g. that if $f(x)=O(1)$, $x \to +\infty$, then $f(x)$ is equal to $\pm \operatorname{Sin} x \pm \cos x$ or $\pm e^{-x}$. (This result also settles question I.)

Just one final remark, and this is to state that one can estimate K in terms of ρ namely, $K \leqq \exp[C_1(1-\rho)^{-1/2}]$; this estimate is furthermore fairly good since for every $\rho<1$ there exists $f(x)$ with $K \geqq \exp[C_2(1-\rho)^{-1/4}]$.

References

1. F. Carlson, *Über Potenzreihen mit ganzzahligen Koeffizienten*, Math. Z. vol. 9 (1921) pp. 1–13.

2. E. C. Titchmarsh, *Theory of functions*, Oxford, 1939, p. 185.

Brown University