A THEOREM CONCERNING SIX POINTS

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1. Introduction and statement of results. "Let three points be specified on a line. Then one of the points is in the interior of the line segment joining the other two, and one of the points is exterior to the line segment joining the other two." This elementary statement concerning the ordering of three points on a line is capable of various extensions. Thus, e.g., it is easy to prove the following: Let \( n + 2 \) points which are not all on an \( n \)-sphere be specified in \( \mathbb{E}^n \), with some \( n + 1 \) of them linearly independent. Then one of the points must be in the interior of the \( n \)-sphere passing through the other \( n + 1 \) points, and one of the points must be exterior to the \( n \)-sphere passing through the other \( n + 1 \) points.

In this paper we consider the following problem: Let six points be specified in a plane, no three collinear and not all on a conic. Must some one of these points be in the "interior" of the conic (i.e. in a convex component of the plane bounded by the conic section) passing through the remaining five, and must some one of the points be "exterior" to the conic (i.e. in the nonconvex component of the plane bounded by the conic) passing through the other five? This question cannot always be answered in the affirmative. We shall see that it is impossible for each point to be outside the conic through the other five, but it is possible for each point to be inside the conic through the other five.

Before stating our result precisely we make the following definition: Let five points no three collinear be specified having either five or three of these points on the boundary \( \beta \) of their convex hull. If \( \beta \) is a pentagon denote its vertices by \( A, B, C, D, E \) taken cyclically about \( \beta \); if \( \beta \) is a triangle denote the two interior points by \( D \) and \( E \) and the other three by \( A, B, C \) such that the half-lines \( DE^+ \) and \( ED^- \) intersect the sides \( AB \) and \( BC \) of \( \beta \) respectively (here and below \( "XY^+" \) designates the (closed) half-line which is bounded by \( X \) and contains \( Y, X \) and \( Y \) being noncoincident points). Then the intersection of the open angular regions \( CAD, DBE, ACE \) each of opening less than \( 180^\circ \) is called the nucleus of the 5-point configuration and denoted by \( N(ABCDE) \).

We first prove the following lemma:
Lemma 1. Let a triangle $T_1(ABC)$ ($A$, $B$, $C$ are the vertices) contain a triangle $T_2(DEF)$ in its interior, no three elements of $T$: $\{A, B, C, D, E, F\}$ collinear. If one of the vertices of $T_2$ is inside the nucleus of the other elements of $T$, then each vertex of $T_2$ is inside the nucleus of the other elements of $T$.

Throughout this paper the symbol $S_6$ will denote a set of six points in a plane ($E_2$), no three of the points collinear and not all on a conic; “$b$” will denote the boundary of the convex-hull of $S_6$.

If one element of $S_6$ is in the interior of the conic through the other five, and one element of $S_6$ is in the exterior of the conic through the other five then we say that $S_6$ is simply-selfcovering. If each element of $S_6$ is in the interior of the conic through the other five then we say that $S_6$ is completely-selfcovering.

We prove the following theorem:

Theorem. (i) If $b$ has 4 or 6 vertices, then $S_6$ is simply-selfcovering.
(ii) If $b$ has 3 or 5 vertices, then $S_6$ is completely- or simply-selfcovering accordingly as any element of $S_6$ not on $b$ belongs to, or does not belong to the nucleus of the other five.

2. Proof of Lemma 1. We let

(i) $l[PQ]$ denote the set of points on the line passing through the points $P$ and $Q$;
(ii) $[PQ]$ denote all points of $l[PQ]$ between (but not including) $P$ and $Q$;
(iii) $[PQR]$ denote the set of all points of the plane interior to (but not including) the triangle with vertices $P$, $Q$, $R$;
(iv) $[PQRS]$ denote the set of all points of the plane interior to (but not including) the simply closed quadrilateral with vertices $P$, $Q$, $R$, $S$ taken cyclically about it;
(v) $\langle PQR \rangle$ ($P$, $Q$, $R$ not collinear) denote the convex open set of all points of the plane bounded by $PQ^{-}$ and $PR^{-}$.

Since no three vertices of $T$ are collinear, we may assume that the vertices of $T_1$ are labelled such that the half-lines $DE^{-}$ and $ED^{-}$ intersect $[AB]$ and $[BC]$ respectively. Also, let $F \in N(ABCDE)$.

2. Since $ED^{-}$ intersects $[BC]$, $D \in [BCE] \subseteq \langle BCE \rangle$.
3. Since $F \in [CEK]$, and $K \in BE^{-}$, $BF^{-}$ separates $D$ and $E$ and intersects $[CE]$ in a point, say $H$. Thus, $D \in [CBH] \subseteq \langle CBF \rangle$.
4. Let $CD^{-}$ and $BD^{-}$ intersect $[BA]$ and $[CA]$ in $Z$ and $Y$ respectively. Since $ED^{-}$ intersects $[BC]$, 

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\[ E \in [ZDYA] \subset [ZCA] = [CDA] + [DA] + [DAZ]. \]

As \([CDEA]\) is strongly convex and can be partitioned thus: \([CDEA]\) = \([CDA]\) + \([DA]\) + \([DEA]\), we have \(E \in [DAZ]\). Since \(F \in [CDA]\), \([AD]\) intersects \([EF]\), and therefore \(D \in \langle EAF \rangle\).

Thus, in view of (1), (2), (3), (4) (above this section), \(D \in N(ABCEF)\).

In similar manner we can prove that \(E \in N(ABCDF)\).

3. "Projectively-cyclic" ordering of five points. Let \(S_5\) denote a set of five points in a (Euclidean) plane \(\pi\), no three of the points being collinear. A conic \(\Gamma\) is uniquely determined by the elements of \(S_5\). Let \(\pi\) be "closed" by adjoining the "line at infinity," and let us denote the projective plane thus obtained by \(\pi_p\). Let \(\Gamma'\) be the conic in \(\pi_p\) such that \(\Gamma' \supset \Gamma\). An ordering \((Q_1, Q_2, Q_3, Q_4, Q_5)\) of \(S_5\) will be referred to as projectively-cyclic if \(\Gamma'\) partitions into abutting but non-overlapping arcs

\[ Q_1Q_2, \ Q_2Q_3, \ Q_3Q_4, \ Q_4Q_5, \ Q_5Q_1. \]

Remark. Thus, e.g. the points

\((-1, 0), \ (-2, 1), \ (1, 0), \ (2, 1), \ (-2, -1)\)

on \(x^2 - y^2 = 1\) are projectively-cyclic in the stated order. If we denote the branches of \(x^2 - y^2 = 1\) by \(B_1\) and \(B_2\), then the element of \(S_5 \cdot B_1\) with largest ordinate is "adjacent" to the element of \(S_5 \cdot B_2\) with smallest ordinate (cf. [1]).

We now show how to order the elements of \(S_5\) so that the ordering is projectively-cyclic on the conic through them.

Let \(\beta\) be the boundary of the convex hull of \(S_5\).

Case I. Suppose \(\beta\) is a pentagon. In this case the points of \(S_5\) are on an ellipse, parabola or one branch of a hyperbola. (For, if they fell on both branches of a hyperbola, \(\beta\) could not be a pentagon.) Then, any cyclic ordering \((Q_1, Q_2, Q_3, Q_4, Q_5)\) of the vertices of \(\beta\) (i.e. \(Q_iQ_{i+1}\) are the edges of \(\beta\), \(Q_6 \equiv Q_1\)) is a projectively-cyclic ordering of \(S_5\).

Case II. Suppose \(\beta\) is a quadrilateral. Then the elements of \(S_5\) obviously cannot fall on an ellipse, parabola or one branch of a hyperbola. Furthermore, it is not possible for one element of \(S_5\) to be on one branch of a hyperbola and the remaining four on the other, for, in this case, \(\beta\) would be a triangle. Thus two elements of \(S_5\) must be on one branch of a hyperbola, and three on the other. Now, let \(O\) be the intersection of the diagonals of \(\beta\); let \(C\) and \(D\) be the adjacent vertices of \(\beta\) such that \([OCD]\) contains the element \(E\) of \(S_5\) which is
not on $\beta$; let $\beta$ be the polygon $(A, B, C, D)$. There are ten ways of partitioning $S_6 = S_2(\alpha) + S_3(\alpha)$ ($i = 1, 2, \ldots, 10$), $S_2(\alpha)$ and $S_3(\alpha)$ containing two and three elements respectively. If for each case we assume that $S_2(\alpha)$ is on one branch of a hyperbola and $S_3(\alpha)$ is on the other, then for nine values of $i$ we obtain a contradiction of the fact that a straight line can cut a conic section in at most two points. The remaining case, where $A$ and $B$ are on one branch and $C, E, D$ on the other, must therefore hold. Furthermore, from the same fact it may be deduced that $E$ is "between" $C$ and $D$ on the branch on which they lie. Thus, the ordering $(A, B, D, E, C)$ (or any cyclic permutation) is projectively-cyclic.

**Case III. Suppose $\beta$ is a triangle.** Then $S_6$ cannot fall on an ellipse, parabola or one branch of a hyperbola; or, on two branches of a hyperbola with two and three points on separate branches. The only remaining case is where one point of $S_6$ is on one branch of a hyperbola and four on the other. Let $X$ and $Y$ be the points of $S_6$ not on $\beta$. Then one of the points of $S_6$, say $P$, must be on one side of the line $l(X, Y)$ and two, say $Q$ and $R$, on the other. Let the half-line $YX$ intersect $\beta$ in the edge $PQ$. Then $P$ is on one branch of a hyperbola and $R, Y, X, Q$ on the other (this is again obtained by considering the five possibilities and eliminating four of them from the fact that a line cuts a conic in at most two points). Furthermore, since the polygon $(R, Y, X, Q)$ is convex, $R, Y, X, Q$ must be the ordering of these points on the branch on which they lie. Thus, the ordering $(P, R, Y, X, Q)$ is projectively-cyclic.

Let $\Delta(Q, R, S, T, U, V)$ be the determinant with $i$th row

$$x_i^2, y_i^2, x_iy_i, x_i, y_i, 1$$

where $(x_i, y_i)$ are the coordinates of the point in the $i$th position from the left in (the parentheses of) $\Delta(\cdot)$.

**Lemma 2.** Let the ordered point set $(W_1, \ldots, W_5)$ (no three collinear) in $E_2$ be projectively-cyclic, and let $\Gamma$ denote the conic through them. Then $\Delta(P, W_1, \ldots, W_5)$ is greater than, equal to or less than zero accordingly as $P$ is inside, on or outside $\Gamma$, and conversely.

**Proof.** Since translations, rotations and proper dilatations ($x' = \alpha x, y' = \gamma y, \alpha > 0, \gamma > 0$) do not alter the sign of $\Delta(P, W_1, \ldots, W_5)$, we may assume that $\Gamma$ is in standard position with (absolute values of) conic constants conveniently chosen. Furthermore, if $W_i$ is moved along $\Gamma$ to $W'_i$ and at no point of this path does $W_i$ cross its adjacent points $W_{i-1}$ or $W_{i+1}$ ($W_0 = W_5$, $W_6 = W_1$) then the sign of $\Delta(P, W_1, \ldots, W_5)$ is unaltered. Thus, if the points $W_1, \ldots, W_5$
are moved along $\Gamma$ to new positions but at no instant does $W_i$ coincide with $W_j$ ($i \neq j$) then the sign of $\Delta(P, W_1, \ldots, W_6)$ remains unaltered. Verification of the sign of $\Delta(P, W_1, \ldots, W_6)$ for simple conics and conveniently chosen points $W_1, \ldots, W_6$ on them, for Cases I, II, III above, completes the proof of the lemma.

**Definition.** Let $S_6 = \{A, B, C, D, E, F\}$. We shall say that $A$ and $B$ are similarly oriented if there are ordered sets $(Z_1, \ldots, Z_4)$ and $(Z_{i_1}, \ldots, Z_{i_4}) (\{Z_1, \ldots, Z_4\} = \{C, D, E, F\})$ each obtainable from the other by an even number of adjacent interchanges such that $(A, Z_1, \ldots, Z_4)$ and $(B, Z_{i_1}, \ldots, Z_{i_4})$ are projectively-cyclic.

**Lemma 3.** If $S_6$ contains two elements which are similarly oriented, then it is simply-self covering.

**Proof.** Let $A$ and $B$ belong to $S_6$ and be similarly oriented. Then $(A, V_1, V_2, V_3, V_4)$ and $(B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4})$ are projectively-cyclic, where $(V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4})$ can be obtained from $(V_1, V_2, V_3, V_4)$ by an even number of adjacent interchanges $(\{A, B, V_1, V_2, V_3, V_4\} = S_6)$. But

\[
\Delta(A, B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}) = \Delta(A, B, V_1, V_2, V_3, V_4) = -\Delta(B, A, V_1, V_2, V_3, V_4).
\]

By Lemma 2 if $A$ is inside the conic $\Gamma_1$ through $B, V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}$ (it can't be on $\Gamma_1$ by definition of $S_6$), then $\Delta(B, A, V_1, V_2, V_3, V_4) < 0$, and $B$ is therefore outside the conic $\Gamma_2$ through $A, V_1, V_2, V_3, V_4$. Similarly, if $A$ is outside $\Gamma_1$, then $B$ is inside $\Gamma_2$.


**Case I.** Suppose $b$ is a hexagon. Then any two adjacent vertices of $b$ are similarly oriented. By Lemma 3, $S_6$ is simply-self covering.

**Case II.** Suppose $b$ is a pentagon ($UVWXZ$). (i) $Z \in N(UVWXZ)$. Then $Z$ is outside a convex quadrilateral $(CDEF), (\{C, D, E, F\} \subset \{U, V, W, X, Y\})$. Let $B \in \{U, V, W, X, Y\}, B \in \{C, D, E, F\}$. Then $B$ and $Z$ are similarly oriented.

(ii) $Z \in N(UVWXZ)$. $Z$ is obviously inside the conic through $U, V, W, X, Y$. We show that $U$ is inside the conic through $V, W, X, Y; Z$: Let $A$ be a point on the half-line $WV^{-}$ with $V$ between $W$ and $A$. Let $J$ be a point on $XY^{-}$ with $Y$ between $X$ and $J$. If $VW^-$ and $YX^{-}$ intersect in a point (or if $l(VW)$ and $l(XY)$ are parallel) then $U$ is inside the (open) convex region bounded by $VA^{-}+ [VY] + YJ^{-}$ which is in the interior of the hyperbola $\Gamma$ passing through $V, Z, Y, W, X$ (cf. Case II, §3). If $VA^{-}$ and $YJ^{-}$ intersect in a point $I$, then $U$ belongs to $[IVY]$ (since $b$ is convex and no three elements
of $S_6$ are collinear) which is in the interior of $\Gamma$. In similar manner it may be shown that $V, W, X$ and $Y$ are each in the interior of the conic through the other five elements of $S_6$.

**Case III.** Suppose $b$ is a quadrilateral $(WXYZ)$. Let the diagonals $[WY]$ and $[XZ]$ intersect in the point $O$.

(i) If $U$ and $V$ are both in the same one of the triangular regions

$\begin{align*}
[OWX], \ [OXY], \ [OYZ], \ [OZW],
\end{align*}$

then they are similarly oriented (cf. Case II of §3).

(ii) Suppose $U$ and $V$ do not fall in the same element of (4.1), and the boundaries of the elements of (4.1) in which they do fall have only the point $O$ in common. Suppose e.g. that $U\in[OZW]$ and $V\in[OXY]$. Then $(U, W, X, Z)$ and $(V, Y, W, Z, X)$ are projectively-cyclic orderings (cf. Case II, §3). But by an even number of adjacent interchanges: $VWZYX \rightarrow VWYZX \rightarrow VWYXZ$. Thus $U$ and $V$ are similarly oriented. The other cases of (ii) are handled similarly.

(iii) Suppose $U$ and $V$ do not fall in the same element of (4.1) and that the boundaries of the elements of (4.1) in which they do fall have an edge in common. Thus, e.g., let $U\in[OWX]$ and $V\in[OXY]$. If the boundary of the convex hull of $W, Z, Y, V, U$ has these five points as its vertices, then $X$ and $Z$ are similarly oriented since $(Z, W, U, V, Y)$ and $(X, W, U, V, Y)$ are projectively-cyclic (cf. Cases I and III of §3). If the boundary of the convex hull of $W, Z, Y, V, U$ has only four vertices e.g. $W, V, Y, Z$, then $Z$ and $X$ are similarly oriented since $(Z, V, U, W, Y)$ and $(X, V, U, W, Y)$ are projectively-cyclic (cf. Cases II and III of §3). The other cases of (iii) are handled similarly.

**Case IV.** Suppose $b$ is a triangle $(XYZ)$. Let $U$ and $V$ be designated so that $U$ and $V$ make with $X$ and $Z$ a convex quadrilateral $(UZXV)$. Let

$\begin{align*}
[ZV] \cdot [XU] = K, & \quad [YU] \cdot [ZV] = L, & \quad [YU] \cdot [ZX] = M, \\
[YV] \cdot [ZX] = N, & \quad [VV] \cdot [UX] = P, & \quad [XV] \cdot [YK] = Q, \\
[ZU] \cdot [YK] = R, & \quad [XV] \cdot [YZ] = S, & \quad [ZU] \cdot [YX] = T.
\end{align*}$

(i) Suppose $W\in N(UVXYZ)$. (a) If $W\in[ZLM]$ then $W$ and $Z$ are similarly oriented. If $W\in[XPN]$ then $W$ and $X$ are similarly oriented. (b) If $W\in[ZKQS]$, then $W$ and $U$ are similarly oriented. If $W\in[XKRT]$, then $W$ and $V$ are similarly oriented. (c) If $W\in[SQY]$, then $W$ and $Y$ are similarly oriented. If $W\in[YRT]$, then $W$ and $Y$ are similarly oriented.
(ii) Suppose $W \in N(UVXYZ)$. We show that each element of $S_6$ is inside the conic through the other five. (a) $W$ is inside the hyperbola $\Gamma$ (cf. Case III, §3) passing through $Z$, $U$, $V$, $X$, $Y$ (since $Z$, $U$, $V$, $X$ are all on one branch of $\Gamma$, $[ZUVX]$ is convex and $Z \in [ZUVX]$). In similar manner using Lemma 1, it may be shown that $U$ and $V$ are each inside the conic through the other five elements of $S_6$. (b) We now show that $Y$ is inside the hyperbola $\theta$ through $X$, $Z$, $U$, $V$, $W$. By Case II of §3, $Z$, $W$, $X$ are on one branch $\theta_1$ of $\theta$ and $U$, $V$ on $\theta_2$ the other branch. Now, $YU^\infty$ intersects $[ZW]$ in a point $G$ which is inside $\theta_1$, where $U$ is between $Y$ and $G$. Therefore $YU^\infty$ intersects $\theta_1$ in a point $H$, where $U$ is between $Y$ and $H$. Thus $Y$ is inside $\theta_2$. In similar manner using Lemma 1, it may be shown that $X$ and $Z$ are each inside the conic through the other five elements of $S_6$.

**Reference**


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