DIFFERENTIABLY SIMPLE RINGS

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Let R be a ring and $\mathfrak{D}$ a family of derivations of R into itself. We call R differentiably simple under $\mathfrak{D}$ if $R^2 \neq 0$ and if R has no two-sided ideal (other than 0 and R) sent into itself under every derivation of the family $\mathfrak{D}$ (i.e., has no differential ideal). We shall call R differentiably simple, usually without specifying $\mathfrak{D}$. The purpose of this paper is to explore the analogy between simple rings and differentiably simple rings.

1. Structure theory.

THEOREM 1. *Let R be differentiably simple; then $R^2 = R$; also, R has no absolute left or right divisors of zero.*

PROOF. R^2 is a nonzero differential ideal, for every derivation. To prove the second part, let for example, $aR = 0$, then $d(aR) = 0 = d(a)R + ad(R) = d(a)R$ so $d(a)R = 0$ so that the absolute left zero divisors are a (two-sided) differential ideal of R; since $R^2 \neq 0$, this ideal is zero.

THEOREM 2. *Let R be differentiably simple and let F be the set of those elements of the centroid of R commuting with every derivation of $\mathfrak{D}$; then F is a field, called the differential centroid of R; if $1 \in R$, F is the subset of R annihilated by every element of $\mathfrak{D}$.*

PROOF. F is contained in the centroid, which is commutative since $R^2 = R$; if b is a nonzero element of F, $b(R)$ is a nonzero differential ideal of R, etc.

LEMMA. *A differentiably simple ring R is not locally nilpotent.*

PROOF. Consider for any a not zero in R the nonzero differential ideal consisting of all sums of two-sided multiples of all products of two derivatives of a of any orders and mixture. This ideal is then the entire ring, so a is in it. Let n be the largest number of derivations occurring in one of the above-mentioned derivatives. We may suppose $n > 0$. Consider the set of derivatives $d_1, \cdots, d_k$ which occur at all in the expression for a; differentiate this expression by every product $d_{i_1} d_{i_2} \cdots d_{i_l}$, $0 \leq l \leq 2n-1$, $1 \leq i_j \leq k$, $1 \leq j \leq l$. Then $d_{i_1} d_{i_2} \cdots d_{i_l}(a) = \sum r_{j_1 \cdots j_q} d_{j_1} \cdots d_{j_q}(a) s_{j_1 \cdots j_q}$ for every $(i_1, \cdots, i_l)$, $0 \leq l \leq 2n-1$ for every $1 \leq i_j \leq k$, the sum being extended over every

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q-tuple $(j_1 \cdots j_q)$, $1 \leqq j_q \leqq k$, $0 \leqq q \leqq 2n-1$. Since the subring of R generated by the $r_{j_1 \dots j_q}$ and $s_{j_1 \dots j_q}$ is nilpotent, by repeated use of this set of equations we find $a=0$ in contradiction to hypothesis.

DEFINITIONS. An ideal divisor of zero in a ring is a two-sided ideal annihilated by a nonzero element, on the right or on the left. A ring is called primary if every ideal divisor of zero is nilpotent.

THEOREM 3. *A differentiably simple ring is primary.*

PROOF. Suppose I is a two-sided ideal of R and $Ib=0$, $b \neq 0$. We shall show that the Levitzki Nil Radical of R contains a differential ideal if $I^n=0$ for no n. $a \in I \Rightarrow axb=0 \forall x \in R$, so $d(a_1)xb+a_1xd(b) =0 \forall x \in R$, $d \in \mathfrak{D}$, $a_1 \in R$. Then $a_2(d(a_1)x)b+a_2a_1xd(b)=0 \forall x \in R$, $a_2a_1xd(b)=0$. In general, $a_na_{n-1} \cdots a_1xd_1d_2 \cdots d_{n-1}(b)=0 \forall a_1, \cdots, a_n \in I$, $d_1, \cdots, d_n \in \mathfrak{D}$. For $a \neq 0$, let $J=\{q/axq=0 \forall x \in R\}$. J is a two-sided ideal of R; we shall show that J is locally nilpotent. For let $K=\{c \in R/cx \cdot \prod_{j=1}^m q_{i_j}=0$, $\forall x \in R$, for all products of any length $m=m(c)$ chosen from a fixed finite set $q_1, \cdots, q_r\}$. K is a two-sided ideal of R, in fact a differential one (where $m(d(c))=2m(c)$), and $K \neq 0$, so $K=R$. A fortiori, every product of length t of the q_l is zero, where $t=2+\max_{1 \leq l \leq r} m(q_l)$. That is, J is locally nilpotent.

Since for any n there is a product of $a_i \in I$, $a_na_{n-1} \cdots a_1$ which is not zero, every mixed derivative of b is in the Levitzki Nil Radical of R (the maximal locally nilpotent ideal of R). R itself would then be locally nilpotent, which contradicts the lemma. So $I^n=0$ for some n, which proves the theorem.

THEOREM 4. *Let R be a differentiably simple ring whose differential centroid is of characteristic zero. Then R is a prime ring, i.e., there are no ideal divisors of zero.*

PROOF. We prove more generally that if R is a primary ring whose additive group is torsion-free, and a lies in a nilpotent ideal, then $d(a)$ lies in a nilpotent ideal, for every derivation d of R. So let n be the smallest integer such that $ax_1ax_2 \cdots ax_n=0 \forall x_i \in R$, $1 \leqq i \leqq n$. Assume $d(a)$ does not lie in a nilpotent ideal. Differentiate the preceding equation n times, and left multiply by $at_1at_2 \cdots at_{n-1}$, where t_j, $1 \leqq j \leqq n-1$ are any elements of R. We obtain $(n+1)at_1at_2 \cdots at_{n-1}d(a)x_1d(a)x_2 \cdots d(a)x_n = 0$, $at_1at_2 \cdots at_{n-1}d(a)x_1d(a)x_2 \cdots d(a)x_n=0 \forall t_j \in R$, $1 \leqq j \leqq n-1$, $\forall x_i \in R$, $1 \leqq i \leqq n$. For simplicity, right multiply by $d(a)$; then $d(a)$ lies in an ideal divisor of zero and hence in a nilpotent ideal, or else $at_1at_2 \cdots at_{n-1}d(a)x_1 \cdots x_{n-1}d(a)=0$. This latter case must occur. Continuing this stripping off process, we find $at_1at_2 \cdots at_{n-1}d(a)=0$, $\forall t_1, \cdots, t_{n-1} \in R$. Then $d(a)$ lies in an ideal divisor of zero, which is impossible, or else $at_1at_2 \cdots at_{n-1} = 0 \forall t_1 \cdots t_{n-1} \in R$, which contradicts the minimality of n.

Then every differentiably simple ring of characteristic zero is prime. For every ideal divisor of zero is nilpotent, so that if R has no nilpotent ideals, we are done; if R has, however, a nilpotent ideal, we have just shown that the sum of the nilpotent ideals of R is a differential ideal, hence all of R. Then R would be locally nilpotent, which we know to be impossible.

For an application of this result to algebraic functions see [1].

COROLLARY. *A differentiably simple ring of characteristic zero with a minimal two-sided ideal is simple. In particular if R satisfies the descending chain condition for left or right ideals, R is simple.*

PROOF. Let R be differentiably simple, and I a minimal 2-sided ideal $\neq 0$. $I^2 \neq 0$ since R is prime, so $I^2 = I$. Then I is differential (for any derivation), so $I = R$, R is simple.

THEOREM 5. *Every commutative differentiably simple ring has a unit.*

PROOF. We must separate the characteristic p and zero cases. First let R be differentiably simple under $\mathfrak{D}$ and of characteristic $p \neq 0$. For every $x \in R$, $d \in \mathfrak{D}$, $d(x^p) = 0$, so (here is where we use commutativity) x^p is in the differential centroid of R. If every $x^p = 0$, then R is a commutative nil ring, hence locally nilpotent. But a differentiably simple ring is not locally nilpotent. Hence for at least one $x \in R$, $x^p \neq 0$ and R contains a nonzero element of the differential centroid, and hence a unit.

Now let R be commutative differentiably simple and of characteristic zero. We shall assume that $\mathfrak{D} = \{d\}$, but only to simplify notation. Look at the proof of the lemma in Theorem 3. Let a be any nonzero element of R. Then there are elements $r_{ij} \in R$ such that $a = \sum_{i=0;j=0}^{n} \sum^{n} r_{ij} a^{(i)} a^{(j)}$. Differentiate $2n-1$ times (without loss of generality, we can assume $n > 0$). Then $a^{(k)} = \sum_{i=0}^{2n-1} t_{k,i} a^{(i)}$, $0 \leqq k \leqq 2n-1$, $t_{k,i} \in R$. Consider the column vector with $2n$ components, v, whose jth component is $a^{(2j-1)}$, and the matrix $T = (t_{k,i})$. Then $v = Tv$. Regard T as a matrix over the quotient field of R (R has no divisors of zero since it is a commutative prime ring). $(T-I)v = 0$. But $v \neq 0$ since $a \neq 0$, $T-I$ is singular.

So $\det(T-I) = 0$. Expand this out. Every term except the product of the 1's on the diagonal contains at least one element of R and the rest 1's. Then $r-(-1)^{2n} \in R$ where $r \in R$ so the unit 1 of the quotient field actually appears as an element of R, R has a unit.

2. Extensions of differentiably simple rings.

THEOREM 6. *Let R be a commutative ring differentiably simple under $\mathfrak{D}$ with differential centroid F. Let M be a nonempty multiplicitively closed subset of R containing* 1 *but not zero. Let R_M be the ring of quotients of R with respect to M. Let $\mathfrak{D}_M$ be the set of derivations of $\mathfrak{D}$ uniquely extended to R_M. Then R_M is differentiably simple under $\mathfrak{D}_M$ with differential centroid F.*

PROOF. Let I be a nonzero differential ideal of R_M; then the set of a in R for which $\exists b \in M$ with $a/b \in I$ is a nonzero differential ideal of R contained in I. So $R \subset I$, $I = R_M$. To prove R_M has differential centroid F, we invoke the following lemma.

LEMMA. *A commutative differentiably simple ring R is differentially closed in its full ring of quotients S.*

PROOF. We are to prove that if $a \in S$ and $d(a) \in R \forall d \in \mathfrak{D}$, then $a \in R$. Let $J = \{x \in R / xa \in R\}$. J is a differential ideal of R and is not zero. Then $J = R$, $1 \cdot a \in R$, $a \in R$.

Returning to the theorem, we are to prove that if $d(a) = 0 \forall d \in \mathfrak{D}$, then $a \in F$. But by the Lemma, $a \in R$, whence $a \in F$.

As an application of this theorem, consider a differential field L. (The differential centroid of a differential field is also called its field of constants.) We wish to prove the known result from differential algebra that an integral can be adjoined to a differential field with the addition of no new constants. Consider L a differential field, x a differential indeterminate over L, and, in the field $L(x)$, extend the original derivation by $x' = a$, where a was not a derivative in L. We wish to show that $L(x)$ has no new constants. However, it can be shown that $L[x]$ is differentiably simple and has the same differential centroid as L. The quotient field $L(x)$ of $L[x]$ has the same differential centroid as $L[x]$, and hence the same as L. That is, $L(x)$ has no new constants.

THEOREM 7. *Let R be differentiably simple under $\mathfrak{D}$ with differential centroid F. Let K be a field containing F. Then $R \otimes_F K$ (under the natural extension of $\mathfrak{D}$) is differentiably simple, with differential centroid K.*

PROOF. Consider the subring of the full ring of F-endomorphisms of R generated by left and right multiplications by elements of R and by differentiation by elements of $\mathfrak{D}$. To say that R is differentiably simple under $\mathfrak{D}$ is the same as saying that R is irreducible under this ring. Furthermore, the differential centroid is just the commuting ring of this ring of endomorphisms. The tensor product over F of this ring and K is the analogous ring for $R\otimes_F K$. Reference [2] is just what we need to say that $R\otimes_F K$ is irreducible under this tensored ring with commuting ring K, i.e., differentiably simple with differential centroid K.

3. **The ring of differential polynomials.** We are going to study the above ring of endomorphisms when R is commutative. First consider the ring of differential polynomials over $\mathfrak{D}$ with coefficients in R, $A_{\mathfrak{D}}(R)$. This will be the ring of polynomials in noncommuting indeterminates, one for each derivation in $\mathfrak{D}$, with coefficients from R written on the left. Here multiplication is defined by $Da=d(a)+aD$ where D is the indeterminate corresponding to the derivation $d\in\mathfrak{D}$. Then $A_{\mathfrak{D}}(R)$ is an (associative) ring, and what is more, R is a left $A_{\mathfrak{D}}(R)$ module by $D(a)=d(a)$. R is contained in $A_{\mathfrak{D}}(R)$ as polynomials of degree zero. Each $d\in\mathfrak{D}$ extends to the inner derivation of $A_{\mathfrak{D}}(R)$ given by $x\to Dx-xD$. R is an irreducible left $A_{\mathfrak{D}}(R)$ module if and only if R is differentiably simple under $\mathfrak{D}$. Let $B_{\mathfrak{D}}(R)$ be $A_{\mathfrak{D}}(R)$ made faithful on R. (This is the ring of the preceding section.) R is still contained in $B_{\mathfrak{D}}(R)$ since R has no absolute left divisors of zero. F is the commuting ring of endomorphisms of $B_{\mathfrak{D}}(R)$ acting on R, and also of $A_{\mathfrak{D}}(R)$ acting on R. Let N be the ideal of $A_{\mathfrak{D}}(R)$ annihilating R, so that $A_{\mathfrak{D}}(R)/N=B_{\mathfrak{D}}(R)$. $B_{\mathfrak{D}}(R)$ is left primitive, R being a faithful irreducible left module.

THEOREM 8. *Let R be a commutative differentiably simple ring of characteristic p ($\neq 0$ forced) of finite dimension over its differential centroid F. Then $B_{\mathfrak{D}}(R)$ is the full ring of F-linear transformations of R into itself. If $R\neq F$, $[R\colon F]\equiv O(p)$; if $\mathfrak{D}$ contains but one derivation, $[R\colon F]$ is a power of p.*

PROOF. $B_{\mathfrak{D}}(R)$ is the full ring of F-linear transformations of R into itself, by the density theorem. Let $R\neq F$, so that $\mathfrak{D}$ contains a nonzero derivation. Consider $\{d(a),\ d\in\mathfrak{D},\ a\in R\}$: there is a non-nilpotent element in this set, since R is not nil. Let $d(a)$ be not nilpotent. Let D be the indeterminate corresponding to d. Then $(Da-aD)^p=(d(a))^p=\lambda\in F$, and λ is a nonzero commutator in $B_{\mathfrak{D}}(R)$, namely $D(a(Da)^{p-1})-(a(Da)^{p-1})D$. The trace of λ is zero, $[R\colon F]\equiv O(p)$. If $\mathfrak{D}=\{d\}$, let k be the least degree of elements of N. Let $J(\subset R)$ be the set consisting of zero and coefficients of D^k for some element of degree k in N. J is an ideal of R, and if $a_0+a_1D+\cdots+a_kD^k\in N$, $D(a_0+a_1D+\cdots+a_kD^k)-(a_0+a_1D+\cdots+a_kD^k)D\in N$ and equals $\cdots+(\cdots)D^{k-1}+d(a_k)D^k$, so that J is a nonzero differential ideal of R. Hence J contains an element $b_0+b_1D+\cdots+b_{k-1}$

$+D^k$. Let $m=[R: F]>1$. $\dim_F A_{\mathfrak{D}}(R)/N=m^2$ by the first part of the theorem, but now $\dim_F A_{\mathfrak{D}}(R)/N\leqq km$ so $m\leqq k$. Since D has a characteristic polynomial of degree $\leqq m$, $k\leqq m$, $k=m$. (Then b_0, $b_1, \cdots, b_{k-1}\in F$ and $X=b_0+b_1D+b_{m-1}D^{m-1}+D^m$ is the characteristic and minimal polynomial for D.) For all $a\in R$, $Xa-aX\in N$ and is of degree less than m, hence is zero.

$$Xa - aX = (X - b_0)(a) + \cdots + [b_{m-1}C_{m-1,1}d(a) + C_{m,2}d^2(a)]D^{m-2} + C_{m,1}d(a)D^{m-1}.$$

Then

$$C_{m,1}D = 0,\ C_{m,1} \equiv O(p); \qquad b_{m-1}C_{m-1,1}D + C_{m,2}D^2 = 0,$$

in particular,

$$C_{m,2} \equiv O(p), \cdots, C_{m,m-1} \equiv O(p).$$

So all nontrivial binomial coefficients of m are multiples of p, m is a power of p, as promised.

We have seen that $B_{\mathfrak{D}}(R)$ is simple if R is finite dimensional over its differential centroid; it is always simple.

THEOREM 9. *If R is a commutative ring differentiably simple under D with differential centroid F, then $B_{\mathfrak{D}}(R)$ is simple with centroid F.*

PROOF. Obviously $(B_{\mathfrak{D}}(R))^2\neq 0$. We must show that if I is a two-sided ideal of $A_{\mathfrak{D}}(R)$, then either I is contained in N, or $I+N=A_{\mathfrak{D}}(R)$. Let x be a nonzero element of I whose degree is minimal with respect to having a nonzero constant term a (if any such exist at all). Then for all b in R, $xb-bx\in I$, is of degree less than x, and has as constant term $(x-a)(b)$. Then $(x-a)(b)=0$ for all $b\in R$, $x-a\in N$, $I+N$ contains the nonzero element a of R. For all $d\in\mathfrak{D}$, $Da-aD\in I+N$, $d(a)\in I+N$, $(I+N)\cap R$ is a nonzero differential ideal and contains 1, $I+N=A_{\mathfrak{D}}(R)$. If on the other hand every element of I has zero constant term, then for all $b\in R$, $xb-bx\in I$ with constant term $x(b)$. Then $x(b)=0$, $x\in N$ if $x\in I$, $I\subset N$.

The above paragraph proves that no element of N has nonzero constant term, for otherwise $A_{\mathfrak{D}}(R)=N$. Let α be in the centroid of $B_{\mathfrak{D}}(R)$ and a, b be in R. $\alpha(ab)=\alpha(a)b, \alpha(ba)=b\alpha(a)$, so $\alpha(a)$ commutes with elements of R. If y in $A_{\mathfrak{D}}(R)$ is $\alpha(a)(N)$, then $yb-by\in N\,\forall b\in R$. But $yb-by$ has constant term $(y-y_0)(b)$, where y_0 is the constant term of y. So $(y-y_0)(b)=0\,\forall b\in R$, $y-y_0\in N$, $\alpha(a)\in R$ in $B_{\mathfrak{D}}(R)$ (being $\equiv$ (mod N) to y_0). That is, by abuse of language, α is in R, since R has a unit. The proof that $d(\alpha)=0\,\forall d\in\mathfrak{D}$ is straightforward and omitted.

REMARK. If F is a commutative differential ring with no absolute divisors of zero such that $B_{\mathfrak{D}}(R)$ is simple, R is differentiably simple. The proof is similar.

THEOREM 10. *Let R be a commutative differentiably simple ring of characteristic zero such that the elements of $\mathfrak{D}$ commute with one another. Let $C_{\mathfrak{D}}(R)$ be the ring of differential polynomials over R in commuting indeterminates. Then $C_{\mathfrak{D}}(R)$ is simple if and only if the elements of $\mathfrak{D}$ are left linearly independent over R.*

The proof is left to the reader.

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