STIRLING NUMBER REPRESENTATION PROBLEMS

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1. **Introduction.** The Stirling numbers of the first kind are defined as the coefficients $S_1(n, k)$ in the expansion

$$\prod_{k=0}^{n} (1 + kx) = \sum_{k=0}^{\infty} S_1(n, k)x^k, \tag{1.1}$$

so that [6] $S_1(n, k) =$ the sum of the $C_{n,k}$ possible products, each with k different factors, which may be formed from the first n natural numbers.

The Stirling numbers of the second kind are defined as the coefficients $S_2(n, k)$ in the expansion

$$\prod_{k=0}^{n} (1 - kx)^{-1} = \sum_{k=0}^{\infty} S_2(n, k)x^k, \tag{1.2}$$

so that $S_2(n, k) =$ the sum of the $C_{(n+k-1),k}$ possible products, each with k factors (repetition allowed), which may be formed from the first n natural numbers.

Schlömilch [9] found the formula

$$\begin{aligned} &(-1)^k S_1(n-1, k) \\ &\quad = (n-k)\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{S_2(j, k)}{(n+j)\binom{k+j}{j}}, \end{aligned} \tag{1.3}$$

which is one of the simplest known explicit representations of the Stirling numbers of the first kind in terms of the Stirling numbers of the second kind. By means of a simple binomial coefficient identity this formula is seen to be equivalent to the neater formula

$$S_1(n-1, k) = \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k+j} S_2(j, k), \tag{1.4}$$

found by L. Schläfli [8].

These two formulas do not seem to be very well known, perhaps because it is easier to calculate S_1 by means of recurrence formulas.

Of course, it is well known [4; 5] that S_2 is given by the very simple formula

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$$(1.5)\qquad \begin{aligned} S_2(n, k) &= \frac{1}{n!} \Delta^n_{x,1} x^{n+k}\Big|_{x=0} \\ &= \frac{(-1)^n}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{n+k}. \end{aligned}$$

We remark that the numbers S_2 occur in the familiar Newton-Gregory expansion [5; 12] of x^n:

$$(1.6)\qquad x^n = \sum_{k=0}^{n} k! S_2(k, n-k) \binom{x}{k}.$$

In this paper we offer simple proofs of the following formulas:

$$(1.7)\qquad (-1)^k S_1(n-1, k) = \binom{n-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \frac{S_2(jn, k)}{\binom{k+jn}{k}},$$

$$(1.8)\qquad (-1)^k S_2(n, k) = \binom{k+n}{k} \sum_{j=0}^{k} (-1)^j \binom{k+1}{j+1} \frac{S_1(jn-1, k)}{\binom{jn-1}{k}},$$

$$(1.9)\qquad S_2(n-k, k) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_1(k+j-1, k),$$

$$(1.10)\qquad S_1(n-1, k) = \sum_{t=0}^{k} K(t) S_1(k+t-1, k),$$

$$(1.11)\qquad S_2(n-k, k) = \sum_{t=0}^{k} K(t) S_2(t, k),$$

where in (1.10) and (1.11)

$$(1.12)\qquad K(t) = \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} \binom{2k+j}{k-t} \binom{-j}{k+t}.$$

In particular we remark that (1.9) is a companion to (1.4) thereby providing a simple way to express the Stirling numbers of the second kind explicitly in terms of the Stirling numbers of the first kind.

We also make application of the Eulerian numbers [1; 12]

$$(1.13)\qquad A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n,$$

in order to show that

$$S_2(n-k,k) = \frac{(-1)^k}{n!}\binom{n}{k}\sum_{i=0}^{k}(-1)^i S_1(k-1,k-i) \cdot \sum_{j=0}^{n} A_{n,j}(j-1)^i. \tag{1.14}$$

2. **Proof of** (1.7) **and** (1.8). Because of the relations

$$\binom{n-1}{k} B_k^{(n)} = (-1)^k S_1(n-1,k), \qquad n \text{ positive integer}, \tag{2.1}$$

and

$$\binom{n+k}{k} B_k^{(-n)} = S_2(n,k), \qquad n \text{ positive integer}, \tag{2.2}$$

where $B_k^{(x)} = B_k^{(x)}(0)$ is a generalized Bernoulli number and [7]

$$\left(\frac{z}{e^z-1}\right)^x \cdot e^{tz} = \sum_{k=0}^{\infty} B_k^{(x)}(t)\frac{z^k}{k!},$$

and also in view of the relations [3]

$$S_1(-n-1,k) = S_2(n,k), \tag{2.3}$$

$$S_2(-n-1,k) = S_1(n,k), \tag{2.4}$$

it will be sufficient to establish for all real n that

$$B_k^{(n)} = \sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1} B_k^{(-jn)}, \tag{2.5}$$

and then (1.7) and (1.8) are special cases.

We take the generalized chain rule of differentiation in the form (cf. [5, p. 216] and [6, p. 22] in general)

$$D_x^k\left(\frac{1}{z}\right) = \sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\frac{1}{z^{j+1}} D_x^k z^j, \tag{2.6}$$

and define

$$z = \left(\frac{e^x-1}{x}\right)^n.$$

Then noting that $\lim_{x\to 0} z = 1$, and that [4; 7]

$$B_k^{(n)} = D_x^k\left(\frac{1}{z}\right)\bigg|_{x=0}, \tag{2.7}$$

we find that (2.5) follows immediately from (2.6).

3. **Proof of** (1.9). We have [7, p. 147]

$$\left(\frac{\log (x+1)}{x}\right)^n = n\cdot \sum_{k=0}^{\infty} \frac{x^k}{k!}\,\frac{B_k^{(n+k)}}{n+k}, \qquad |x| < 1. \tag{3.1}$$

From this and [7, p. 145]

$$B_k^{(n+1)}(1) = \left(1 - \frac{k}{n}\right) B_k^{(n)}, \tag{3.2}$$

it follows that

$$\left(\frac{x}{\log (x+1)}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!} B_k^{(k-n+1)}(1). \tag{3.3}$$

Now the generalized chain rule may also be written in the convenient form (cf. [5, p. 216] and [6, p. 22])

$$z^n D_x^k z^{-n} = \sum_{j=0}^{k} \binom{-n}{j}\binom{k+n}{k-j} z^{-j} D_x^k z^j, \tag{3.4}$$

and by an easy binomial coefficient identity this may also be written as

$$(-1)^k \binom{n-1}{k} z^n D_x^k z^{-n} = \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j}\binom{k+j}{j} z^{-j} D_x^k z^j. \tag{3.5}$$

We define

$$z = \frac{\log (x+1)}{x}$$

and note that $\lim_{x\to 0} z = 1$. Then it follows from (3.5) and the expansions (3.1) and (3.3) together with (3.2) that

$$(-1)^k \binom{n}{k} B_k^{(k-n)} = \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j}\binom{k+j-1}{k} B_k^{(j+k)}, \tag{3.6}$$

and consequently when we apply (2.2) to the left-hand member and (2.1) to the right-hand member, this expression becomes identically (1.9) which is therefore proved.

4. **Proof of** (1.10) **and** (1.11). It is a routine calculation to substitute for $S_2(j, k)$ in (1.4) by means of (1.9) and obtain (1.10). Likewise we substitute for $S_1(k+j-1, k)$ in (1.9) by means of (1.4) and the result is exactly (1.11). The summation $K(t)$ occurs in each case.

5. **Proof of** (1.14). Worpitzky [**12**, formula (14)] has shown that

$$k!S_2(k, n-k) = \sum_{j=0}^{n} \binom{j-1}{n-k} A_{n,j}, \tag{5.1}$$

where $A_{n,j}$ are the Eulerian numbers defined by (1.13).

Now it is a consequence of (1.1) that the familiar expansion

$$\binom{x}{n} = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{n!} S_1(n-1, n-k)x^k \tag{5.2}$$

is obtained.

From (5.1) we obtain, first putting $n-k$ for k and then using (5.2),

$$\begin{aligned}(n-k)!S_2(n-k, k) &= \sum_{j=0}^{n} \binom{j-1}{k} A_{n,j} \\ &= \sum_{j=0}^{n} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{k!} S_1(k-1, k-i)(j-1)^i A_{n,j}.\end{aligned}$$

Simplification of this yields relation (1.14) as proposed.

It would be interesting to obtain a relation inverse to (1.14), that is a formula expressing S_1 in terms of S_2 using $A_{n,j}$.

It is not hard to show that a relation inverse to (5.1) is

$$(-1)^k A_{n,k} = \sum_{j=0}^{n} (-1)^j \binom{n-j}{n-k} j!S_2(j, n-j). \tag{5.3}$$

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