VERIFICATION OF A CONJECTURE OF GERSTENHABER

J. P. JANS

Let T_n denote the algebra of matrices (A_{ij}) where $A_{ij}=0$ for $i\leqq j$ and $A_{ij}\in F$, a field. In a recent paper [1], M. Gerstenhaber conjectured that T_n/T_n^k could not be represented by $n\times n$ matrices if $n\geqq 5$ and $3\leqq k\leqq n-1$. He proved there that the conjecture is true for $k=n-1$. The purpose of this note is to verify this conjecture in general. In fact, we shall prove something a bit stronger.

THEOREM. *If M is a faithful T_n/T_n^k module where $3\leqq k\leqq n-1$ then* $[M: F]\geqq 3+(k-2)(n-k+1)$.

PROOF. Fix k and n and denote by T the algebra T_n/T_n^k. If M is a faithful T-module then we have the descending sequence $M\supset TM \supset T^2M\supset \cdots \supset T^{k-1}M\supset T^kM=0$. We remark that if S is a set of vectors in M then to show S independent it is enough to show that the vectors of S in $T^{i-1}M$ not in T^iM are independent for $i=1, \cdots, k$. We shall use this principle in the proof.

The algebra T has a basis of matrix units E_{ij} subject to the condition $k-1\geqq i-j\geqq 1$, with the multiplication table

$$E_{ij}E_{km} = \delta_{jk}E_{im} \qquad \text{if } i-m\leqq k-1,$$
$$E_{ij}E_{km} = 0 \qquad \text{if } i-m> k-1.$$

For fixed j, $1\leqq j\leqq n-k+1$, there is a left ideal L_j spanned by the elements E_{ij}, $i=j+1, \cdots, j+k-1$ (each of these left ideals has dimension $k-1$ over F).

In the faithful T-module M we shall find submodules M_j corresponding to the L_j. Each of the M_j will have dimension k over F. We will not be able to show that the sum of the M_j is direct, but we will be able to show a certain amount of independence between them. In this way we can see that the dimension of M is large enough.

Since M is faithful there exists x_j in M such that $E_{j+k-1\,j}\,x_j\neq 0$. Thus the vectors x_j, $E_{j+1\,j}\,x_j, \cdots, E_{j+k-1\,j}\,x_j$ are independent and span a submodule M_j of M. That they are independent follows from the remark at the beginning of the proof.

We claim that the following set of vectors is independent:

(*) x_1, $E_{k1}x_1$, and $E_{ij}x_j$ for $1\leqq i-j\leqq k-2$ and $1\leqq j\leqq n-k+1$.

In this set there are $2+(k-2)(n-k+1)$ vectors. After we show that these are independent, we will find one more vector independent of all the vectors in (*). That would complete the proof of the theorem.

Received by the editors August 31, 1959.

We note that x_1 is in M but not in TM and all the rest of the vectors in (*) are in TM. Also $E_{k1}x_1$ is in $T^{k-1}M$ and none of the other vectors in (*) are in $T^{k-1}M$. The vectors of (*) which are in T^tM and not in $T^{t+1}M$ are of the form $E_{t+r\,r}x_r$ for $r=1, \cdots, n-k+1$ and $1 \leqq t \leqq k-2$. If there is a dependence relation of the form

$$\sum_{r=1}^{n-k+1} \beta_r E_{t+r\,r}\, x_r = 0,$$

then the equation can be multiplied by matrix basis elements of the form $E_{i+1\,i}$. The condition, $t \leqq k-2$, insures that every term but one then drops out and each of the $\beta_r = 0$.

Thus by the remark at the beginning of the proof the set (*) is independent. We wish to select one more vector x' independent of (*). If x_2 is not a multiple of x_1 then $x_2 = x'$ will do the job because x_1 is the only vector of (*) not in TM. If $E_{k+1\,2}x_2$ is not a multiple of $E_{k1}x_1$, then we can let $x' = E_{k+1\,2}x_2$ since $E_{k1}x_1$ is the only vector of (*) in $T^{k-1}M$.

Now suppose that $x_2 = \alpha x_1$ and $E_{k+1\,2}x_2 = \beta E_{k1}x_1$ where α and β are both not zero. But then $(\alpha E_{k+1\,2} - \beta E_{k1})x_1 = 0$ and there exists a vector x' such that $(\alpha E_{k+1\,2} - \beta E_{k1})x' \neq 0$. The vectors x_1 and x' are independent because $(\alpha E_{k+1\,2} - \beta E_{k1})$ annihilates one and not the other and neither x_1 nor x' is in TM. Thus, in any case, we can find a vector x' independent of (*) and the dimension of M over the field F is at least $3 + (k-2)(n-k+1)$.

We remark that the nilpotent algebras T_n considered here are closely related to the algebras of matrices S_n consisting of matrices (A_{ij}) such that $A_{ij} = 0$ if $i < j$. Let S be S_n factored by the kth power of its radical. S is almost the algebra T of the theorem. It can be shown that the algebra S has $n-k+1$ k-dimensional left ideals (corresponding to the ideals L_j in the proof of the above theorem) which are projective injective modules over S. Thus, by a theorem in [2] every faithful S-module must contain as a direct summand the direct sum of these left ideals. Then any faithful S-module has dimension at least $k(n-k+1)$. It can be shown that, for S, this inequality cannot be improved. However, we think the inequality of the theorem could be improved a bit by replacing the "3" by something depending on n and k (possibly k itself).

References

1. M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices*, III, Ann. of Math. vol. 70 (1959) pp. 167–205.

2. J. P. Jans, *Projective injective modules*, Pacific J. Math. vol. 9 (1959) pp. 1103–1108.

University of Washington