VERIFICATION OF A CONJECTURE OF GERSTENHABER

J. P. JANS

Let $T_n$ denote the algebra of matrices $(A_{ij})$ where $A_{ij} = 0$ for $i \leq j$ and $A_{ij} \in F$, a field. In a recent paper [1], M. Gerstenhaber conjectured that $T_n/T_n^k$ could not be represented by $n \times n$ matrices if $n \geq 5$ and $3 \leq k \leq n - 1$. He proved there that the conjecture is true for $k = n - 1$. The purpose of this note is to verify this conjecture in general. In fact, we shall prove something a bit stronger.

**Theorem.** If $M$ is a faithful $T_n/T_n^k$ module where $3 \leq k \leq n - 1$ then $[M: F] \geq 3 + (k - 2)(n - k + 1)$.

**Proof.** Fix $k$ and $n$ and denote by $T$ the algebra $T_n/T_n^k$. If $M$ is a faithful $T$-module then we have the descending sequence $M \supset T^2M \supset \cdots \supset T^{k-1}M \supset T^k M = 0$. We remark that if $S$ is a set of vectors in $M$ then to show $S$ independent it is enough to show that the vectors of $S$ in $T^{i-1}M$ not in $T^i M$ are independent for $i = 1, \cdots, k$. We shall use this principle in the proof.

The algebra $T$ has a basis of matrix units $E_{ij}$ subject to the condition $k - i = j - 1$, with the multiplication table

\[
\begin{array}{c|c}
E_{ij}E_{km} & \delta_{jk}E_{im} \\
\hline
E_{ij}E_{km} & 0 \\
\end{array}
\]

if $i - m \leq k - 1$,

if $i - m > k - 1$.

For fixed $j$, $1 \leq j \leq n - k + 1$, there is a left ideal $L_j$ spanned by the elements $E_{ij}$, $i = j, j + 1, \cdots, j + k - 1$ (each of these left ideals has dimension $k - 1$ over $F$).

In the faithful $T$-module $M$ we shall find submodules $M_j$ corresponding to the $L_j$. Each of the $M_j$ will have dimension $k$ over $F$. We will not be able to show that the sum of the $M_j$ is direct, but we will be able to show a certain amount of independence between them. In this way we can see that the dimension of $M$ is large enough.

Since $M$ is faithful there exists $x_j$ in $M$ such that $E_{j+k-1}x_j \neq 0$. Thus the vectors $x_j$, $E_{j+1}x_j$, $\cdots$, $E_{j+k-1}x_j$ are independent and span a submodule $M_j$ of $M$. That they are independent follows from the remark at the beginning of the proof.

We claim that the following set of vectors is independent:

\[(*) \ x_1, E_{k+1}x_1, \text{ and } E_{ij}x_j \text{ for } 1 \leq i - j \leq k - 2 \text{ and } 1 \leq j \leq n - k + 1.\]

In this set there are $2 + (k - 2)(n - k + 1)$ vectors. After we show that these are independent, we will find one more vector independent of all the vectors in $(*)$. That would complete the proof of the theorem.

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We note that \( x_1 \) is in \( M \) but not in \( TM \) and all the rest of the vectors in (*) are in \( TM \). Also \( E_{k1}x_1 \) is in \( T^{k-1}M \) and none of the other vectors in (*) are in \( T^{k-1}M \). The vectors of (*) which are in \( T^tM \) and not in \( T^{t+1}M \) are of the form \( E_{+1+r}x_r \) for \( r = 1, \ldots, n-k+1 \) and \( 1 \leq t \leq k-2 \). If there is a dependence relation of the form

\[
\sum_{r=1}^{n-k+1} \beta_r E_{+1+r}x_r = 0,
\]

then the equation can be multiplied by matrix basis elements of the form \( E_{+1} \). The condition, \( t \leq k-2 \), insures that every term but one then drops out and each of the \( \beta_r = 0 \).

Thus by the remark at the beginning of the proof the set (*) is independent. We wish to select one more vector \( x' \) independent of (*). If \( x_2 \) is not a multiple of \( x_1 \) then \( x_2 = x' \) will do the job because \( x_1 \) is the only vector of (*) not in \( TM \). If \( E_{k+1}x_2 \) is not a multiple of \( E_{k1}x_1 \), then we can let \( x' = E_{k+1}x_2 \) since \( E_{k1}x_1 \) is the only vector of (*) in \( T^kM \).

Now suppose that \( x_2 = \alpha x_1 \) and \( E_{k+1}x_2 = \beta E_{k1}x_1 \) where \( \alpha \) and \( \beta \) are both not zero. But then \( (\alpha E_{k+1} - \beta E_{k1})x_1 = 0 \) and there exists a vector \( x' \) such that \( (\alpha E_{k+1} - \beta E_{k1})x' \neq 0 \). The vectors \( x_1 \) and \( x' \) are independent because \( (\alpha E_{k+1} - \beta E_{k1}) \) annihilates one and not the other and neither \( x_1 \) nor \( x' \) is in \( TM \). Thus, in any case, we can find a vector \( x' \) independent of (*) and the dimension of \( M \) over the field \( F \) is at least \( 3 + (k-2)(n-k+1) \).

We remark that the nilpotent algebras \( T_n \) considered here are closely related to the algebras of matrices \( S_n \) consisting of matrices \( (A_{ij}) \) such that \( A_{ij} = 0 \) if \( i < j \). Let \( S \) be \( S_n \) factored by the \( k \)th power of its radical. \( S \) is almost the algebra \( T \) of the theorem. It can be shown that the algebra \( S \) has \( n-k+1 \) \( k \)-dimensional left ideals (corresponding to the ideals \( L_j \) in the proof of the above theorem) which are projective injective modules over \( S \). Thus, by a theorem in \([2]\) every faithful \( S \)-module must contain as a direct summand the direct sum of these left ideals. Then any faithful \( S \)-module has dimension at least \( k(n-k+1) \). It can be shown that, for \( S \), this inequality cannot be improved. However, we think the inequality of the theorem could be improved a bit by replacing the "3" by something depending on \( n \) and \( k \) (possibly \( k \) itself).

**References**