SOME APPLICATIONS OF AN APPROXIMATION THEOREM FOR INVERSE LIMITS[1]

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Introduction. In 1954 C. E. Capel proved [1] the following theorems: *Let S be the inverse limit of a sequence of arcs* (*simple closed curves*) *where the bounding maps are onto and monotone. Then S is an arc* (*simple closed curve*). It may be noted that if f is a monotone map of an arc (simple closed curve) onto itself, then f is the uniform limit of a sequence of onto homeomorphisms.[2] We call such a map a *near-homeomorphism.* In this paper we prove the following two theorems: (1) *If S is the inverse limit of a sequence of copies of a given compact metric space X and the bonding maps are near-homeomorphisms, then S is homeomorphic to X.* (2) *Let $f: X \to Y$, $g: Y \to X$, where f, g are maps and X, Y are compact metric spaces. Suppose fg and gf are near-homeomorphisms. Then X is homeomorphic to Y.* The second theorem follows directly from the first. In order to establish the first theorem we develop an approximation theorem which has interest in its own right.

Definitions and Notation. Let X_i be a sequence of compact metric spaces, and for $i \geqq 2$ let f_i map X_i into X_{i-1}. Then the subspace[3] $S = \{z \in \prod_1^\infty X_i \mid f_{ij}(z_j) = z_i\}$ of $\prod_1^\infty X_i$ is the *limit space* of the inverse system (X_i, f_i); in notation $S = \text{Lim}(X_i, f_i)$.

Let f map X into Y where X, Y are compact metric spaces. Then for $\epsilon > 0$: $L(\epsilon, f) = \text{Sup}\{\delta < d(X) \mid x, y \in X$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon\}$.[4] Since X is compact, $0 < L(\epsilon, f) \leqq d(X)$.

Let (X_i, f_i) be an inverse system. A sequence (a_i) of positive real numbers is a *Lebesgue sequence* for (X_i, f_i) if there is a sequence (b_i) of positive real numbers such that: (1) $\sum b_i < \infty$. (2) Whenever $x, y \in X_j$, $i < j$, and $|x-y| < a_j$, then $|f_{ij}(x) - f_{ij}(y)| < b_j$. A sequence (c_i) of positive real numbers is a *measure* for (X_i, f_i) if: (1) $\sum_{n+1}^\infty c_i < (1/2)c_n$, $n = 1, 2, \cdots$, (2) for any two points s, s' of Lim (X_i, f_i) there is an integer n such that $|s_{n+1} - s'_{n+1}| > c_n$.

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[2] A simple proof for the case of an arc is the following: Suppose f: $[01] \to [01]$ is monotone. Obviously we may assume that $f(0) = 0$, and $f(1) = 1$. Let h_n: $[01] \to [01]$ by $h_n(x) = (1 - 1/n)f(x) + (1/n)x$.

[3] $f_{ij} = f_{i+1} f_{i+2} \cdots f_j$, $f_{ii} = 1$. If z is a point of X_i then z_i will always denote the ith coordinate of z. Hence $z = (z_i)$.

[4] $d(X)$ denotes the diameter of X. $|x-y|$ denotes the distance from x to y.

LEMMA 1. *Let* $S=\mathrm{Lim}(X_i, f_i)$ *where the* X_i *are compact metric. Then* (X_i, f_i) *has a Lebesgue sequence* (a_i).

PROOF. Let $a_j=\mathrm{Min}_{i<j}\, L(2^{-i}, f_{ij})$, $b_i=2^{-i}$. Then $\sum b_i<\infty$, and whenever $x, y\in X_j$, $i<j$, and $|x-y|<a_j$, then $|x-y|<L(2^{-i}, f_{ij})$. Hence $|f_{ij}(x)-f_{ij}(y)|<2^{-i}$.

LEMMA 2. *Let* $S=\mathrm{Lim}(X_i, f_i)$ *where the* X_i *are compact metric. Then* (X_i, f_i) *has a measure* (c_i).

PROOF. Let (a_i) be the Lebesgue sequence constructed in the proof of Lemma 1. Let $c_1=(1/2)a_2$, $c_{i+1}=(1/2)\ \mathrm{Min}[(1/2)c_i,\ a_{i+2}]$. Then (c_i) is the required measure. For suppose s, s' are distinct points of S. Then there exist positive integers n, j with $n<j$, and such that $|s_n-s_n'|>2^{-i}$. If $|s_j-s_j'|\leqq c_{j-1}$ then $|s_j-s_j'|<a_j$. Hence $|s_n-s_n'|<2^{-j}$ which is absurd. Hence $|s_j-s_j'|>c_{j-1}$.

THEOREM 1. *Let* $S=\mathrm{Lim}(X_i, f_i)$, $T=\mathrm{Lim}(X_i, g_i)$ *where the* X_i *are compact metric spaces. Suppose* $\|f_{i+1}-g_{i+1}\|<a_i$, $i=1, 2, \cdots$, *where* (a_i) *is a Lebesgue sequence for* (X_i, g_i). *Then the function* $F_N\colon S\to X_N$ *defined by* $F_N=\mathrm{Lim}_n\, g_{Nn}f_{n\,\infty}$ *is well defined and continuous.*[5] *Moreover, the function* $F\colon S\to T$ *defined by* $F(s)=(F_1(s), F_2(s), \cdots,)$ *is well defined, continuous, and onto.*

PROOF. Let N be fixed. Let (b_i) be the sequence associated with (a_i).

$$\text{(1.1)}\qquad \lim_{i\to\infty;\, N<i<j} \|g_{Ni}f_{ij} - g_{Ni}g_{ij}\| = 0.$$

PROOF OF (1.1). $\|g_{Ni}f_{ij}-g_{Ni}g_{ij}\| = \sum_{r=i}^{j-1}\|g_{Ni}g_{ir}f_{rj}-g_{Ni}g_{i\,r+1}f_{r+1\,j}\|$. Now $g_{Ni}g_{ir}f_{rj}=g_{Nr}f_{r+1}f_{r+1\,j}$, and $g_{Ni}g_{i\,r+1}f_{r+1\,j}=g_{Nr}g_{r+1}f_{r+1\,j}$. Since $\|g_{Nr}f_{r+1}f_{r+1\,j} - g_{Nr}g_{r+1}f_{r+1\,j}\| \leqq \|g_{Nr}f_{r+1} - g_{Nr}g_{r+1}\|$, $\|g_{Ni}f_{ij} - g_{Ni}g_{ij}\| \leqq \sum_{r=i}^{j-1}\|g_{Nr}f_{r+1}-g_{Nr}g_{r+1}\|$. But the hypothesis assures that $\|f_{r+1}-g_{r+1}\|<a_r$. Hence $\|g_{Nr}f_{r+1}-g_{Nr}g_{r+1}\|<b_r$. Hence $\|g_{Ni}f_{ij}-g_{Ni}g_{ij}\|\leqq\sum_{r=i}^{j-1} b_r \leqq \sum_{r=1}^{\infty} b_r$. Finally $\mathrm{Lim}_{i\to\infty}\|g_{Ni}f_{ij}-g_{Ni}g_{ij}\|\leqq\mathrm{Lim}_{i\to\infty}\sum_{r=i}^{\infty} b_r=0$.

$$\text{(1.2)}\qquad \lim_{i\to\infty;\, i<j} \|g_{Ni}f_{i\,\infty} - g_{Nj}f_{j\infty}\| = 0.$$

The proof of (1.2) precisely parallels that of (1.1).

$$\text{(1.3)}\qquad \lim_{i\to\infty} g_{Ni}f_{i\,\infty}(s) = F_N(s) \text{ uniformly in } s.$$

(1.3) follows immediately from (1.2).

(1.4) Hence $F_N=\mathrm{Lim}_{i\to\infty}\, g_{Ni}f_{i\,\infty}$ is well defined and continuous.

[5] $f_{n\,\infty}$ is the map projecting each point of S onto its nth coordinate.

From this it follows immediately that $F\colon S\to\prod_{i=1}^{\infty} X_i$ is well defined and continuous.

(1.5) $F(S)\subset T$. To prove (1.5) it suffices to show that

$$g_{ij}F_j(s) = F_i(s) \qquad \text{for all } s\in S,\ i<j.$$

$$\begin{aligned} g_{ij}F_j(s) &= g_{ij}\lim_{n\to\infty} g_{jn}f_{n\,\infty}(s) \\ &= \lim_{n\to\infty} g_{ij}g_{jn}f_{n\,\infty}(s) \\ &= \lim_{n\to\infty} g_{in}f_{n\,\infty}(s) \\ &= F_i(s). \end{aligned}$$

(1.6) $F\colon S\to T$ is onto.

Proof of (1.6). Let $t=(t_n)\in T$. Let N be fixed. We first show that there is an $s^N\in S$ such that $F_N(s^N)=t_N$. Let $\epsilon>0$. By (1.1), (1.3), and the convergence of $\sum_{i=1}^{\infty} b_i$, there is an $i>N$ such that $\|F_N-g_{Ni}f_{i\,\infty}\|<\epsilon/3$, $\|g_{Ni}f_{ij}-g_{Ni}g_{ij}\|<\epsilon/3$ all $j>i$; and $b_i<\epsilon/3$. Fix this i. Now by Corollary 3.8, Chapter VIII of [2] $\bigcap_{j=i}^{\infty} f_{ij}(X_j)=f_{i\,\infty}(S)$. Since the X_j are compact there is a $j>i$ such that each point of $f_{ij}(X_j)$ is of distance less than a_i from a point of $f_{i\,\infty}(S)$. Hence there is a point $s\in S$ such that $|f_{ij}(t_j)-f_{i\,\infty}(s)|<a_i$. Hence $|g_{Ni}f_{ij}(t_j)-g_{Ni}f_{i\,\infty}(s)|<b_i<\epsilon/3$. Then

$$\begin{aligned} |F_N(s)-t_N| &\leqq |F_N(s)-g_{Ni}f_{i\,\infty}(s)| + |g_{Ni}f_{i\,\infty}(s)-g_{Ni}f_{ij}(t_j)| \\ &\quad + |g_{Ni}f_{ij}(t_j)-g_{Ni}g_{ij}(t_j)| \\ &\leqq \epsilon/3+\epsilon/3+\epsilon/3=\epsilon. \end{aligned}$$

Hence the compactness of S insures the existence of an $s^N\in S$ such that $F_N(s^N)=t_N$. Now for all N, $F_N(s^N)=t_N$ implies that $F_i(s^N)=t_i$ for $i<N$. If $s\in S$ is a convergence point of the set $\{s^N\}$ then $F(s)=t$.

Remark. F is not necessarily 1-1. For suppose $T=\operatorname{Lim}(I_i, g_i)$ where I_i is the unit interval $[01]$, and $g_i(t)=0$, $t\in I_i$. Let $S=\operatorname{Lim}(I_i, f_i)$ where $f_i(t)=t$, $t\in I_i$. Let $a_i=2$ for all i. Then (a_i) is a Lebesgue sequence for (I_i, g_i) where $b_i=2^{-i}$. For if $x, y\in I_j$ then $|g_{ij}(x)-g_{ij}(y)|=0<2^{-i}$. Also $|f_{i\,i+1}(t)-g_{i\,i+1}(t)|$ is t, and $t<a_i$. Since S is an arc and T is a point, F cannot be 1-1.

Theorem 2. *Let* $S=\operatorname{Lim}(X_i, f_i)$, $T=\operatorname{Lim}(X_i, g_i)$ *where the* X_i *are compact metric spaces. Suppose*

$$\|f_i-g_i\| < \min\left[c_{i-1};\ \min_{k<i-1} L(c_{i-1}, g_{k\,i-1})\right]$$

where (c_i) is a measure for (X_i, f_i). Then the map $F: S \to T$ described in Theorem 1 is a homeomorphism onto.

PROOF. To prove that F is well defined continuous and onto, it suffices to show that $a_i = \min_{k<i} L(c_i, g_{ki})$ is a Lebesgue sequence for (X_i, g_i). The sequence we associate with (a_i) is (c_i). Since $c_N > 2\sum_{N+1}^{\infty} c_i$, $\sum_1^{\infty} c_i < \infty$. Finally, suppose $x, y \in X_j$, $i<j$, and $|x-y| < a_j$. Then $|x-y| < L(c_j, g_{ij})$. Hence $|g_{ij}(x) - g_{ij}(y)| < c_j$.

$$(2.1) \qquad \|f_{Ni} - g_{Ni}\| < \sum_{j=N}^{\infty} c_j, \qquad \text{all } N, i, (N < i).$$

PROOF OF (2.1).

$$\begin{aligned}\|g_{Ni} - f_{Ni}\| &\leqq \|g_{Ni-1}g_i - g_{Ni-1}f_i\| + \|g_{Ni-1}f_i - f_{Ni-1}f_i\|,\\ &\leqq \|g_{Ni-1}g_i - g_{Ni-1}f_i\| + \|g_{Ni-1} - f_{Ni-1}\|.\end{aligned}$$

Since $\|g_i - f_i\| < a_{i-1}$, $\|g_{Ni-1}g_i - g_{Ni-1}f_i\| < c_{i-1}$. Hence

$$\|g_{Ni} - f_{Ni}\| < c_{i-1} + \|g_{N\,i-1} - f_{N\,i-1}\|.$$

Continuing recursively, $\|g_{Ni} - f_{Ni}\| < c_{i-1} + \cdots + c_N$.

$$(2.2) \qquad F \text{ is } 1-1.$$

PROOF OF (2.2). Suppose $F(s) = F(s')$, $s, s' \in S$. If $s \neq s'$, then for some $n > 1$, $c_{n-1} < |s_n - s_n'|$. Now for $i > n$,

$$\begin{aligned}c_{n-1} &< |s_n - s_n'| = |f_{ni}(s_i) - f_{ni}(s_i')|\\ &\leqq |f_{ni}(s_i) - g_{ni}(s_i)| + |g_{ni}(s_i) - g_{ni}(s_i')| + |g_{ni}(s_i') - f_{ni}(s_i')|.\end{aligned}$$

Applying (2.1) we get:

$$(2.3) \qquad c_{n-1} < 2\sum_{j=n}^{\infty} c_j + |g_{ni}(s_i) - g_{ni}(s_i')|, \qquad \text{all } i > n.$$

Letting i approach infinity, $|g_{ni}(s_i) - g_{ni}(s_i')| \to |F_n(s) - F_n(s')| = 0$. Hence (2.3) becomes $c_{n-1} \leqq 2\sum_{j=n}^{\infty} c_j$. But this contradicts the requirement that $c_{n-1} > 2\sum_{j=n}^{\infty} c_j$.

THEOREM 3. *Let $S = \mathrm{Lim}(X_i, f_i)$ where the X_i are compact metric spaces. For $i \geqq 2$ let K_i be a nonempty collection of maps from X_i into X_{i-1}. Suppose that for each $i \geqq 2$ and $\epsilon > 0$ there is $g \in K_i$ such that $\|f_i - g\| < \epsilon$. Then there is a sequence (g_i) where $g_i \in K_i$ and S is homeomorphic to $\mathrm{Lim}(X_i, g_i)$.*

PROOF. Let (c_i) be a measure for (X_i, f_i). Let g_2 be any element of K_2. Let g_3 be an element of K_3 such that $\|f_3 - g_3\| < L(c_2, g_2)$. Inductively, let g_{n+1} be an element of K_{n+1} such that $\|f_{n+1} - g_{n+1}\|$

$<\min_{k<n} L(c_n, g_{kn})$. (Here $g_{kn}=g_{k+1}g_{k+2}\cdots g_n$). Then by Theorem 2 if $T=\text{Lim}(X_i, g_i)$, $F\colon S\to T$ homeomorphically onto.

Near homeomorphisms. Let X be a metric space. A map $f\colon X\to X$ is a *near homeomorphism* if for any $\epsilon>0$ there is a homeomorphism H_ϵ of X onto itself such that $\|H_\epsilon-f\|<\epsilon$.

THEOREM 4. *Let $S=\text{Lim}\,(X_i, f_i)$ where the X_i are all homeomorphic to a compact metric space X, and for all i, f_i is a near homeomorphism. Then S is homeomorphic to X.*

PROOF. By Theorem 3 there is a sequence h_i of homeomorphisms of X_i onto X_{i-1} such that $S=\text{Lim}(X_i, h_i)$. But now S must be homeomorphic to X.

COROLLARY. *Let $S=\text{Lim}\,(X_i, f_i)$ where each X_i is a copy of a fixed 2-manifold X (compact and with or without boundary) and f_i is monotone onto. Then S is homeomorphic to X.*

PROOF. J. W. T. Youngs [4] has proven that if $f\colon X\to X$ is monotone onto and X is a compact 2 manifold (with or without boundary) then f is a near homeomorphism.

THEOREM 5. *Let X, Y be compact metric spaces. Suppose f maps X into Y, g maps Y into X, and fg and gf are near homeomorphisms. Then X is homeomorphic to Y.*

PROOF. The following diagram commutes:

$$\begin{array}{ccccccccccccccc}
X & & \xleftarrow{\quad gf \quad} & & X & & \xleftarrow{\quad gf \quad} & & X & & \xleftarrow{\quad gf \quad} & & X & \leftarrow & \cdots S_1 \\
\downarrow 1 & & & & \downarrow 1 & & & & \downarrow 1 & & & & \downarrow 1 & & \\
X & \xleftarrow{g} & Y & \xleftarrow{f} & X & \xleftarrow{g} & Y & \xleftarrow{f} & X & \xleftarrow{g} & Y & \xleftarrow{f} & X & \leftarrow & \cdots S_2 \\
& & \downarrow 1 & & & & \downarrow 1 & & & & \downarrow 1 & & & & \\
& & Y & & \xleftarrow{\quad fg \quad} & & Y & & \xleftarrow{\quad fg \quad} & & Y & & \longleftarrow & & \cdots S_3.
\end{array}$$

Hence S_1, S_2, and S_3 are mutually homeomorphic. But by Theorem 4 S_1 is homeomorphic to X and S_3 is homeomorphic to Y.

COROLLARY. *Let X, Y be compact metric spaces. Suppose f maps X onto Y, g maps Y into X, gf is a near homeomorphism. Then* $\text{Dim}(X)\leqq\text{Dim}(Y)$.

PROOF. Examination of the diagram for Theorem 5 yields that X can be expressed as the inverse limit of Y's. Hence $\text{Dim}(X)\leqq\text{Dim}(Y)$.

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NEW METHOD FOR EXPANSION AND CONTRACTION MAPS IN UNIFORM SPACES

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1. Introduction and definitions. In [2], Freudenthal and Hurewicz showed that if the function f, from the totally bounded metric space M onto M, has the property that $(fx, fy) \leqq (x, y)$ for each x and y in M, then f is an isometry. By amplifying the sequential argument given in [2], Rhodes (see [4]) proved that an even stronger result holds in the more general setting of uniform spaces. Using a different method, the present paper offers a theorem similar to that of Rhodes, together with a number of results concerning "expansion" maps in uniform spaces. The notation used here, which very closely approximates that of [4], has been taken from Chapter 6 of [3].

1.1. Definition. If $(M, \mathfrak{U})$ is a uniform space, then a subset $\mathfrak{B}$ of $\mathfrak{U}$ will be called a basis for $(M, \mathfrak{U})$ if

(a) if $x \in M$ and $U \in \mathfrak{B}$, then $(x, x) \in U$;

(b) if $U \in \mathfrak{U}$, then U^{-1} contains a member of $\mathfrak{B}$;

(c) for each $U \in \mathfrak{U}$ there is a $V \in \mathfrak{B}$ for which $V \circ V \subset U$; and

(d) for each $U \in \mathfrak{U}$ and $V \in \mathfrak{U}$, there is a $W \in \mathfrak{B}$ for which $W \subset U \cap V$.

1.2. Definition. If $\mathfrak{B}$ is a basis for the uniform space $(M, \mathfrak{U})$, then $\mathfrak{B}$ is said to be open if each of its elements is open in $M \times M$.

1.3. Definition. If $\mathfrak{B}$ is a basis for the uniform space $(M, \mathfrak{U})$, then $\mathfrak{B}$ is said to be ample if, whenever $(x, y) \in U \in \mathfrak{B}$, there is a $W \in \mathfrak{B}$ for which $(x, y) \in W \subset \overline{W} \subset U$.

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