CONDITIONS FOR THE POWER ASSOCIATIVITY OF ALGEBRAS

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In this paper we extend the results of Albert [1] and Kokoris [2; 3] to obtain conditions for the power associativity of algebras over arbitrary fields in the noncommutative case. The following results of Albert [1] are used extensively in the development of the theory.

LEMMA 1. *Let A be an algebra over a field of characteristic p, $p\neq 2$, where $x^a x^b = x^{a+b}$ for $a+b<n$, $n\geqq 4$, $x\in A$. Then for $c=1, 2, \cdots, n-1$ it follows that* $[x^{n-c}, x^c] = c[x^{n-1}, x]$, $n[x^{n-1}, x]=0$ *and for* $(n, p)=1$ *also* $x^{n-c}x^c = x^c x^{n-c}$.

LEMMA 2. *Let A be an algebra over a field of characteristic p, $p\neq 2$, 3 or 5, where $x^a x^b = x^{a+b}$ for $a+b<n$, $n\geqq 5$, $x\in A$. Then for $c=1, 2, \cdots, n-1$ it follows that $x^{n-c}x^c = x^n$ for $(n, p)=1$ and $x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for all n.*

LEMMA 3.[2] *Let A be an algebra over a field of characteristic p, $p\neq 2$, where $x^a x^b = x^{a+b}$ for $a+b<n=kp^r$, $k\neq 1$, $(k, p)=1$, r any positive integer, $x\in A$. Then $[x^{n-1}, x]=0$.*

PROOF. We show that $[x^{n-k}, x^k]=0$, then by Lemma 1 have $k[x^{n-1}, x]=0$ and $(k, p)=1$ yields the desired result. $x^{n-k}x^k = x^{kp^r-k}x^k = (x^k)^{p^r-1}x^k = x^k(x^k)^{p^r-1} = x^k x^{n-k}$, using the assumption that $k\neq 1$ and that powers with less than n factors associate.

In the results that follow the reader will easily note that the results of [1; 2; 3] for commutative algebras are consequences of the general theorems.

CASE 1, $p\neq 2$, 3 or 5.

THEOREM 1. *Let A be an algebra over a field of characteristic p, $p\neq 2$, 3 or 5, and such that for all $x\in A$ and all positive integers r,* (i) $x^2x = xx^2$, (ii) $x^3x = x^2x^2$, (iii) $x^{p^r-1}x = xx^{p^r-1}$. *Then A is power associative.*

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[2] This is a slight extension of Lemma 1 and Lemma 2, not in [1].

PROOF. We observe from Lemma 1 that $x^3x=xx^3$ so that all powers with less than five factors associate. Assuming associativity for all powers with less than n factors, $n\geqq 5$, we consider three cases:

(i) For $(n, p)=1$, Lemma 2 yields $x^{n-c}x^c=x^n$.

(ii) For $n=p^r$, hypothesis (iii) and Lemma 2 yield the result.

(iii) For $n=kp^r$, $k\neq 1$, $(k, p)=1$, Lemma 3 yields $[x^{n-1}, x]=0$ and Lemma 2 then gives the result.

CASE 2, $p=5$. From Albert [1] the identities $x^2x=xx^2$ and $x^3x=x^2x^2$ imply, for $3\leqq a+b+c<n$,

$$\begin{aligned}&3(x^{n-a}x^a + x^{n-b}x^b + x^{n-c}x^c + x^{n-(a+b+c)}x^{a+b+c})\\ &= 4(x^{n-a-b}x^{a+b} + x^{n-a-c}x^{a+c} + x^{n-b-c}x^{b+c}) - (a+b+c)[x^{n-1}, x].\end{aligned} \tag{2.1}$$

Assuming associativity for powers with less than n factors and setting $a=b=1$, $c=2$ in (2.1) yields

$$x^{n-4}x^4 = x^{n-3}x^3 + 2x^{n-2}x^2 + 3x^{n-1}x + 2[x^{n-1}, x]. \tag{2.2}$$

Under the same associativity conditions, letting the triple (a, b, c) have the values (1, 1, 3), (1, 1, 4) and (2, 2, 2) we obtain the following three identities which combine with (2.2) to yield (2.3).

$$\begin{aligned}x^{n-5}x^5 &= x^{n-4}x^4 + 4x^{n-3}x^3 + 3x^{n-2}x^2 + 3x^{n-1}x,\\ x^{n-6}x^6 &= x^{n-5}x^5 + 4x^{n-4}x^4 + 3x^{n-2}x^2 + 3x^{n-1}x + 3[x^{n-1}, x],\\ x^{n-6}x^6 &= 4x^{n-4}x^4 + 2x^{n-2}x^2 + 3[x^{n-1}, x].\end{aligned}$$

$$x^{n-2}x^2 = x^{n-1}x + 3[x^{n-1}, x] \qquad \text{for } n \geqq 7. \tag{2.3}$$

LEMMA 4. *Let A be an algebra over a field of characteristic 5 and let $x^2x=xx^2$, $x^3x=x^2x^2$, $x^4x=xx^4$, $x^5x=x^4x^2$ for $x\in A$. Then $x^ax^b=x^{a+b}$ for $a+b<7$, and assuming associativity in powers of less than n factors, $n\geqq 7$, it follows that $x^{n-c}x^c=x^n$ for $(n, 5)=1$ and $x^{n-c}x^c=x^{n-1}x+((c-1)/2)[x^{n-1}, x]$ for $c<n$.*

PROOF. By Lemma 1, $x^3x=xx^3$ and a substitution of 5 for n in (2.2) with Lemma 1 yields $x^3x^2=x^4x$ so associativity holds for all powers with three, four or five factors. Lemma 1 gives commutativity of sixth powers, and a substitution of 6 for n in (2.1) with $a=b=c=1$ yields $2x^4x^2=4x^5x+3x^3x^3$ which, together with the hypothesis $x^5x=x^4x^2$, establishes associativity in sixth powers. The case $c=1$ of $x^{n-c}x^c=x^{n-1}x+((c-1)/2)[x^{n-1}, x]$ is trivial and $c=2$ holds by (2.3). Assuming the validity for $c=1, 2, \cdots, k-1$ and letting $a=k-2$, $b=c=1$ in (2.1) we find $x^{n-k}x^k=x^{n-k+1}x^{k-1}+4x^{n-k+2}x^{k-2}+3x^{n-2}x^2+3x^{n-1}x+(k/2)[x^{n-1}, x]=x^{n-1}x+((k-1)/2)[x^{n-1}, x]$. Then for $(n, 5)=1$, $x^{n-c}x^c=x^n$ follows by Lemma 1.

THEOREM 2. *Let A be an algebra over a field of characteristic 5 and let $x^2x=xx^2$, $x^3x=x^2x^2$, $x^5x=x^4x^2$ and $x^{5^r-1}x=xx^{5^r-1}$ for all $x\in A$ and all positive integers r. Then A is power associative.*

PROOF. Following the pattern of proof of Theorem 1, an obvious induction in three cases using the hypotheses and Lemmas 3 and 4 yields the proof.

CASE 3, $p=3$. In the case $p=3$ we make the restriction that the base field is not the prime field and proceed in the same fashion. Letting $a=k-1$ and $b=c=1$ in (2.1) we find for $1<k<n-1$

$$x^{n-k}x^k = x^{n-2}x^2 + ((k+1)/2)[x^{n-1}, x]. \tag{3.1}$$

Thus to get a result corresponding to Lemmas 2 and 5 we relate $x^{n-2}x^2$ and $x^{n-1}x$ in terms of the commutator $[x^{n-1}, x]$. A substitution of $x+\lambda y$ for x in $xx^4-x^2x^3=0$ yields a polynomial equation in λ, $A\lambda+B\lambda^2+C\lambda^3+D\lambda^4=0$, and since the base field has at least four nonzero elements $A=0$. This is just

$$x[x^3y + (x^2y)x + ((xy+yx)x)x] + yx^4 = x^2[x^2y + (xy+yx)x] + (xy+yx)x^3.$$

Assuming associativity for all powers with less than n factors, $n\geqq 6$, and setting $y=x^{n-4}$ we find,

$$x^{n-4}x^4 = 2x^{n-3}x^3 + 2xx^{n-1}.$$

Now putting $a=b=2$, $c=1$, and $a=3$, $b=c=1$ in (2.1) and combining yields,

$$x^{n-4}x^4 = 2x^{n-3}x^3 + 2x^{n-2}x^2.$$

The last two identities and the definition of the commutator yield

$$x^{n-2}x^2 = xx^{n-1} = x^{n-1}x + 2[x^{n-1}, x]. \tag{3.2}$$

Combining (3.1) and (3.2) we have for $1<k<n-1$

$$x^{n-k}x^k = x^{n-1}x + ((k-1)/2)[x^{n-1}, x]. \tag{3.3}$$

LEMMA 5. *Let A be an algebra over a field of characteristic 3, not the prime field, and let $x^2x=xx^2$, $x^3x=x^2x^2$, $x^4x=x^3x^2$ for all $x\in A$. Then $x^ax^b=x^{a+b}$ for $a+b<6$ and assuming that all powers with less than n factors associate, $n\geqq 6$, it follows that $x^{n-c}x^c=x^n$ for $(n, 3)=1$ and $x^{n-c}x^c=x^{n-1}x+((c-1)/2)[x^{n-1}, x]$ for $c<n$.*

PROOF. By Lemma 1, with $n=4$ and 5, and the hypotheses we have associativity for powers with less than 6 factors. Identity (3.3) is

$x^{n-c}x^c = x^{n-1}x + ((c-1)/2)[x^{n-1}, x]$ for $c=2, 3, \cdots, n-2$ and the case $c=1$ is trivial. For $c=n-1$ we observe that $x^{n-(n-1)}x^{n-1} = xx^{n-1} = x^{n-1}x - [x^{n-1}, x]$ and use $n[x^{n-1}, x]=0$ to write $xx^{n-1} = x^{n-1}x - [x^{n-1}, x] + (n/2)[x^{n-1}, x] = x^{n-1}x - (((n-1)-1)/2)[x^{n-1}, x]$ as desired. $x^{n-c}x^c = x^n$ for $(n, 3)=1$ then follows from Lemma 1.

THEOREM 3. *Let A be an algebra over a field of characteristic 3, not the prime field, and let $x^3x = x^2x^2$, $x^4x = x^3x^2$ and $x^{3^r-1}x = xx^{3^r-1}$ for all $x \in A$ and all positive integers r. Then A is power associative.*

PROOF. As before, the results follow from the hypotheses by Lemmas 3 and 5, using an induction in three cases.

CASE 4, $p=2$. In this case we make the restriction that the base field is not the prime field and proceed in a manner somewhat similar to that of Kokoris [3]. We substitute $x+\gamma y$ for x in $x^2x - xx^2 = 0$ to obtain $A\gamma + B\gamma^2 = 0$ where $A = (xy+yx)x + x^2y - [x(xy+yx) + yx^2]$. Since the base field has at least two nonzero elements, $A=0$, and letting $y = x^a$ and assuming associativity for powers with less than $a+2$ factors we obtain,

$$x^2x^a = x^ax^2. \tag{4.1}$$

A substitution of $x+\lambda y$ for x in $x^3x - x^2x^2 = 0$ gives the polynomial $A\lambda + B\lambda^2 + C\lambda^3 = 0$ where $B = 0 = (x^2y)y + ((xy+yx)x)y + ((xy+yx)y)x + (y^2x)x - [x^2y^2 + (xy+yx)(xy+yx) + y^2x^2]$. Setting $y = x^a$ in $B=0$, assuming associativity for powers with less than $2a+2$ factors, and using (4.1) yields the first of the following identities. The second follows from $xx^3 - x^2x^2 = 0$ by a parallel argument.

$$x^{a+2}x^a = x^{2a+1}x, \qquad x^ax^{a+2} = xx^{2a+1}. \tag{4.2}$$

Replacing y by $x^a + x^{n-(a+2)}$ in $B=0$ and assuming associativity for powers with less than n factors we obtain the first of the following results. The second is the parallel identity.

$$x^{n-a}x^a = x^{a+2}x^{n-(a+2)}, \qquad x^ax^{n-a} = x^{n-(a+2)}x^{a+2}. \tag{4.3}$$

From (4.3) we have immediately,

$$x^{n-a}x^a + x^ax^{n-a} = x^{n-(a+2)}x^{a+2} + x^{a+2}x^{n-(a+2)} \tag{4.4}$$

and by a simple induction (4.3) also yields for $0 \leqq 4t \leqq n-2$,

$$x^{n-1}x = x^{n-(4t+1)}x^{4t+1}. \tag{4.5}$$

LEMMA 6. *Let A be an algebra over a field of characteristic 2, not the prime field, and let $x^ax^b = x^{a+b}$ for $a+b<n$, $n \geqq 5$. Then $x^{n-2}x^2 = x^{n-b}x^b$ for b even, $0<b<n$, and $x^{n-1}x = x^{n-a}x^a$ for n odd and all a, $0<a<n$.*

PROOF. From (4.1) and (4.4) we obtain $x^{n-b}x^b = x^b x^{n-b}$ for all even b. Then (4.3) yields $x^{n-c}x^c = x^{n-(c+2)}x^{c+2}$ for all even c and by induction $x^{n-2}x^2 = x^{n-b}x^b$ for all even b, $0 < b < n$. When n is odd, $n-1$ is even so the above argument yields $x^{n-1}x = xx^{n-1}$ which extends by (4.4) to $x^{n-c}x^c = x^c x^{n-c}$ for odd c, $0 < c < n$. By (4.3) $x^{n-1}x = x^{n-c}x^c$. For even c we know that $x^{n-c}x^c = x^{n-2}x^2 = x^2 x^{n-2}$, but n is odd, $n-2$ is odd and $x^2 x^{n-2} = x^{n-1}x$ by our last result, completing the proof.

LEMMA 7. *Under the hypotheses of Lemma* 6, $x^{n-1}x = x^{n-a}x^a$ *for* a *odd*, $0 < a < n$, n *even, and also* $x^{n-1}x = x^{n-a}x^a$ *for all* a, $0 < a < n$, $n = 2k$ *with* k *odd and* $k > 1$.

PROOF. We consider two cases, $n = 2^r$ and $n = 2^r k$, k odd, $k > 1$. In the former case the hypothesis $x^3 x = xx^3$ gives $x^{n-1}x = x^{n-a}x^a$ for a odd, $r < 3$, so we consider only $r \geqq 3$. Letting $t = 2^{r-3}$ in (4.5) and $a = 2^{r-1} - 1$ in the second relation of (4.2) we obtain $x^{2^r-1}x = xx^{2^r-1}$. This extends by (4.3) and (4.4) to $x^{2^r-1}x = x^{2^r-a}x^a$ for a odd, $0 < a < n = 2^r$. In the latter case we note that $x^{2^r k - k}x^k = (x^k)^{2^r-1}x^k = (x^k)(x^k)^{2^r-1} = x^k x^{2^r k - k}$. Using (4.4) this is extended to $x^{n-1}x = x^{n-a}x^a$ for all odd a less than n. To establish the second conclusion of the lemma we exhibit an even b, $0 < b < n = 2k$ such that $x^{n-1}x = x^{n-b}x^b$, then Lemma 6 and the above complete the proof. $k-1$ serves as this b as is seen by substitution of $k-1$ for a in the first of (4.2).

THEOREM 4. *Let* A *be an algebra over a field of characteristic* 2, *not the prime field, and let* $x^2 x = xx^2$, $x^3 x = x^2 x^2 = xx^3$ *and* $x^{2^r-1}x = x^{2^{r-1}}x^{2^{r-1}}$ *for all* $x \in A$ *and all integers* $r > 2$. *Then* A *is power associative.*

PROOF. By hypotheses we have associativity for all powers with less than five factors. Assuming associativity for powers with less than n factors, $n \geqq 5$, consider the following cases:

(i) n odd or $n = 2k$ for k odd, $k > 1$; Lemma 6 or 7 establishes the induction.

(ii) $n = 2^r$; hypothesis $x^{2^{r-1}}x^{2^{r-1}} = x^{2^r-1}x$ yields $x^{n-1}x = x^{n-b}x^b$ for the even $b = 2^{r-1}$ and as in the preceding we have associativity for powers with n factors.

(iii) $n = 2^r k$ for k odd, $k > 1$; $x^{2^{r-1}k}x^{2^{r-1}k} = (x^k)^{2^{r-1}}(x^k)^{2^{r-1}} = x^{2^r k - k}x^k$ where $2^{r-1}k$ is even and k is odd so that, as above, the induction is completed.

AN EXAMPLE. Examples which show that the conditions of these theorems cannot be weakened appear in Albert [1] and Kokoris [3] with the exception of the hypothesis $x^2 x = xx^2$ in all cases and the hypothesis $x^{p^r-1}x = xx^{p^r-1}$ in Theorems 1, 2 and 3. The first example is easily constructed after Albert [1] and that for the second set of conditions is again very similar to one of Albert [1] but we outline the construction since the computation is rather involved.

Let A be the algebra with basis $[a, a^2, \cdots, a^{p^n-1}, a^{p^n-1}a, aa^{p^n-1}]$ over a field F of characteristic p, $p\neq 0$ or 2 and $p^n>5$. Define a product by $a^s a^t = a^{s+t}$ for $s+t<p^n$, $a^s a^t = a^{p^n-1}a + 2^{-1}(t-1)[a^{p^n-1}, a]$ for $s+t=p^n$ and requiring any product with more than p^n factors to be zero. Albert [1] has shown that for $p\neq 3, 5$ this algebra satisfies the hypotheses of Theorem 1 except $x^{p^i-1}x = xx^{p^i-1}$ for $i<n$, and the extension to Theorems 2 and 3 is straightforward. For the remaining condition we write the general element of A as

$$x = \sum_{i=1}^{p^n-1} \lambda_i a^i + \lambda_{p^n} a^{p^n-1}a + \lambda_{p^n+1} aa^{p^n-1} \qquad \lambda_k \in F.$$

Making use of the multinomial expansion

$$x^{p^r-1} = \sum_{k=p^r-1}^{p^n-1} \left[\sum_e \frac{(p^r-1)!}{\prod (e_i!)} \prod \lambda_i^{e_i} \right] a^k + ta^{p^n-1}a + yaa^{p^n-1}$$

where $k=\sum ie_i$, e_i are the elements of the partition e of p^r-1, and the values of t and y will not concern us. Since all powers with less than p^n factors associate we may write

$$x^{p^r-1}x = \sum_m \left[\sum_f \frac{p^r!}{\prod (f_i!)} \prod \lambda_i^{f_i} \right] a^m + \sum_j \left[\sum_e \frac{(p^r-1)!}{\prod (e_i)!} \prod \lambda_i^{e_i}\lambda_j \right] a^k a^j$$

where $m=\sum if_i$, f_i are the elements of the partition f of p^r, and the k and e_i are as above with $k+j=p^n$. We obtain a similar expression for xx^{p^r-1} and compute the difference

$$x^{p^r-1}x - xx^{p^r-1} = \sum_e \sum_j j \frac{(p^r-1)!}{\prod (e_i)!} \prod \lambda_i^{e_i}\lambda_j [a^{p^n-1}, a].$$

However, $\prod \lambda_i^{e_i}\lambda_j = \prod \lambda_i^{f_i}$ since $f_i=e_i$ for $i\neq j$ and $f_j=e_j+1$ so that

$$\frac{j}{\prod (e_i!)} = \frac{jf_j}{\prod (f_i!)}$$

and since $\sum jf_j = p^n$ the difference becomes

$$\sum_e \frac{p^r!}{\prod (f_i!)} p^{n-r} \prod \lambda_i^{f_i} [a^{p^n-1}, a].$$

For any $r<n$ these coefficients are zero and we have then $x^{p^r-1}x = xx^{p^r-1}$ for all $r<n$.

AN APPLICATION. A noncommutative Jordan algebra has been defined as an algebra satisfying the identities $x(yx)=(xy)x$ and $(x^2y)x = x^2(yx)$ and R. D. Schafer has proved that any such algebra over a field of characteristic not two is power associative. We extend this result to the characteristic two case when the field is not the prime field. The identities $x^2x=xx^2$ and $x^3x=x^2x^2=xx^3$ are trivial and a substitution of $y=x^{2^n-3}$ yields $x^{2^n-1}x=x^2x^{2^n-2}$ under the assumption that all powers with less than 2^n factors associate. Since 2^n-2 and 2^{n-1} are even Lemma 6 yields $x^{2^n-1}x=x^{2^{n-1}}x^{2^{n-1}}$ and the proof of Theorem 4 applies directly.

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