DIVISIBLE MODULES

EBEN MATLIS

Introduction. Let \( R \) be an integral domain with quotient field \( Q \), and let \( A \) be a module over \( R \). \( A \) is said to be a divisible \( R \)-module, if \( rA = A \) for every \( r \neq 0 \in R \). An element \( x \in A \) is said to be a torsion element of \( A \), if there exists \( r \neq 0 \in R \) such that \( rx = 0 \). The set of torsion elements of \( A \) is a submodule of \( A \) called the torsion submodule of \( A \), and we will consistently denote it by \( AT \). We will let \( E(A) \) denote the injective envelope of \( A \) (see [3]); and \( \text{hd}_R A \) will denote the homological dimension of \( A \) as an \( R \)-module.

We will study conditions, some necessary, some sufficient, for the torsion submodule of a divisible module to be a direct summand. These will be related to the condition that \( \text{hd}_Q Q = 1 \), where \( Q \) is the quotient field of \( R \). We will apply these conditions to show that, if \( R \) is a Noetherian integral domain in which prime ideals different from zero are maximal, and if \( D \) is a divisible module over \( R \), then \( D \) is a homomorphic image of an injective \( R \)-module and \( DT \) is a direct summand of \( D \). The same conclusions hold, if we merely assume for an arbitrary integral domain that its quotient field is countably generated as a module over the ring.

1. The torsion submodule. It is easy to see that, if \( C \) is an injective module over an integral domain \( R \), then its torsion submodule \( CT \) is also an injective \( R \)-module, and therefore a direct summand. Namely, any homomorphism of an ideal of \( R \) into \( CT \) can be extended to a homomorphism of \( R \) into \( C \); but this extension must in fact map \( R \) into \( CT \). The following theorem is a generalization of this fact.

Theorem 1.1. Let \( R \) be an integral domain and \( H \) a homomorphic image of an injective \( R \)-module. Then \( HT \) is a direct summand of \( H \).

Proof. The mapping of a free \( R \)-module \( F \) onto an injective \( R \)-module \( C \) can be extended to a mapping of \( E(F) \) onto \( C \). We can thus assume that there exists a torsion-free, divisible \( R \)-module \( U \) and an epimorphism \( f: U \to H \). Now \( H/HT \), being torsion-free and divisible, is a direct sum of \( R \)-modules \( Q_i \), where \( Q_i \) is isomorphic to \( Q \) the quotient field of \( R \). Let \( S_i \) be the inverse image of \( Q_i \) under the canonical map \( H \to H/HT \). We will prove that \( HT \) is a direct summand of each \( S_i \), and by [2, Lemma 2] this will complete the proof of the theorem.

Presented to the Society, September 3, 1959; received by the editors May 23, 1959.
Let \( y \in S_j - H_T \); then \( Ry \) is a torsion-free submodule of \( S_j \). Choose \( x \in U \) such that \( f(x) = y \). Since \( U \) is a vector space over \( Q \), there exists an \( R \)-submodule \( T_j \) of \( U \) such that \( T_j \cong Q \) and \( x \in T_j \). If \( u \neq 0 \in T_j \), there exist \( r, s \in R \) such that \( ru = sx \neq 0 \). Since \( rf(u) = sf(x) = sy \neq 0 \), we have \( f(u) \neq 0 \), and thus \( f(T_j) \cong T_j \cong Q \).

Since \( H = \sum_i S_i \), \( f(u) = w + z \), where \( w \in \sum_i S_i \) and \( z \in S_j \). Hence \( r \) \( w + rz = rf(u) = sy \). Thus \( rw + rz = rf(u) = sy \in S_j \). Therefore, \( w \in H_T \subset S_j \), and so \( f(u) \in S_j \). This shows that \( f(T_j) \subset S_j \). Since \( f(T_j) \cong Q \), we have \( f(T_j) \cap H_T = 0 \) and \( f(T_j) \) maps onto \( Q_j \) under the canonical map \( H \to H/H_T \). Thus \( S_j = H_T \oplus f(T_j) \); and so \( H_T \) is a direct summand of \( H \).

**Theorem 1.2.** Let \( R \) be an integral domain with quotient field \( Q \neq R \). Suppose that \( D_T \) is a direct summand of \( D \) for every divisible \( R \)-module \( D \). Then \( hd_R Q = 1 \).

**Proof.** Let \( A \) be any \( R \)-module and let \( E = E(A) \). Since \( E \) is an essential extension of \( A \), \( E/A \) is a torsion \( R \)-module. Let \( G \) be any \( R \)-module extension of \( E/A \) by \( Q \). Since both \( Q \) and \( E/A \) are divisible, \( G \) is also divisible. Clearly \( E/A \) is the torsion submodule of \( G \); and thus by assumption \( E/A \) is a direct summand of \( G \). Thus \( Ext^1_R(Q, E/A) = 0 \) [1, Theorem 14.1.1]. We also have \( Ext^n_R(Q, E) = 0 \) for \( n > 0 \), since \( E \) is injective. Therefore, from the exact sequence:

\[
\begin{align*}
\text{Ext}^1_R(Q, E/A) & \to \text{Ext}^2_R(Q, A) \to \text{Ext}^2_R(Q, E)
\end{align*}
\]

we deduce that \( \text{Ext}^2_R(Q, A) = 0 \). Hence \( \text{hd}_R Q \leq 1 \). Since \( Q \) is not \( R \)-projective, \( \text{hd}_R Q = 1 \).

Over an arbitrary integral domain it is not true that \( D_T \) is a direct summand for every divisible module \( D \). For by the above theorem this would imply that \( \text{hd}_R Q = 1 \). However, I. Kaplansky has shown (unpublished) that \( \text{hd}_R Q = 1 \) for a valuation ring \( R \) if and only if \( Q \) is a countably generated \( R \)-module.

**Theorem 1.3.** Let \( R \) be an integral domain with quotient field \( Q \neq R \), and suppose that \( Q \) is countably generated as a \( R \)-module. Then every divisible \( R \)-module \( D \) is a homomorphic image of an injective \( R \)-module. Thus \( D_T \) is a direct summand of \( D \), and \( \text{hd}_R Q = 1 \).

**Proof.** There exists a countable set of generators \( \{q_n\} \) for \( Q \) over \( R \), and elements \( \{a_{n+1}\} \) of \( R \) such that \( q_1 = 1 \) and \( a_{n+1}q_{n+1} = q_n \). Let \( D \) be a divisible \( R \)-module, and let \( x \neq 0 \in D \). We define a mapping \( f \) from the generators \( \{q_n\} \) to \( D \). Let \( f(1) = x \); then there exists \( x_2 \in D \) such that \( a_2x_2 = x \), and we define \( f(q_2) = x_2 \). There exists \( x_3 \in D \) such that \( a_3x_3 = x_2 \), and we define \( f(q_3) = x_3 \). We continue in this way and
define $f$ on all the generators $\{q_n\}$. It is easily verified that $f$ induces an $R$-homomorphism from $Q$ into $D$ such that the image contains $x$. It is now clear that by taking a big enough direct sum $G$ of copies of $Q$ we can define an $R$-homomorphism of $G$ onto $D$.

It should be remarked that if $R$ is any integral domain and $S$ a countable, multiplicatively closed subset of $R$, then it can be easily shown that if $F$ is a countably generated free $R$-module and $f$ a suitably chosen mapping of $F$ onto $R_S$, then the kernel of $f$ is free; and thus $\text{hd}_R R_S \leq 1$.

2. $\text{hd}_R Q = 1$.

**Proposition 2.1.** Let $R$ be an integral domain with quotient field $Q$ such that $\text{hd}_R Q = 1$. Let $H$ be an $R$-module. Then the following statements are equivalent:

1. $\text{Ext}^1_R(Q/R, H) = 0$.
2. Every $R$-homomorphism from $R$ into $H$ can be extended to an $R$-homomorphism from $Q$ into $H$.
3. $H$ is a homomorphic image of an injective $R$-module.

**Proof.** That (1) implies (2) follows immediately from the exact sequence:

$$\text{Hom}_R(Q, H) \to \text{Hom}_R(R, H) \to \text{Ext}^1_R(Q/R, H).$$

That (2) implies (3) is trivial. That (3) implies (1) follows from the fact that $\text{hd}_R Q/R = 1$.

**Proposition 2.2.** Let $R$ be an integral domain with quotient field $Q$ such that $\text{hd}_R Q = 1$. Let $H$ be an $R$-module. Then:

1. If $H$ is a homomorphic image of an injective $R$-module, so is $H_T$.
2. If $B$ is a submodule of $H$ and if $B$ and $H/B$ are homomorphic images of injective modules, then so is $H$.

**Proof.**

1. If $H$ is a homomorphic image of an injective $R$-module, then $H_T$ is a direct summand of $H$ by Theorem 1.1. Hence $H_T$ is also a homomorphic image of an injective $R$-module.
2. Suppose that $B$ and $H/B$ are homomorphic images of injective $R$-modules. We have an exact sequence:

$$\text{Ext}^1_R(Q/R, B) \to \text{Ext}^1_R(Q/R, H) \to \text{Ext}^1_R(Q/R, H/B).$$

By Proposition 2.1 the two end modules are zero, and thus $\text{Ext}^1_R(Q/R, H) = 0$. Hence by Proposition 2.1 again, $H$ is a homomorphic image of an injective $R$-module.
DEFINITION. Let $B$ be a module over an integral domain. Then we will say that $B$ is $h$-reduced, if $B$ has no nonzero submodules which are homomorphic images of injective modules.

**Corollary 2.3.** Let $A$ be a module over an integral domain $R$ with quotient field $Q$ such that $\text{hd}_R Q = 1$. Then $A$ has a unique largest submodule $H$ which is a homomorphic image of an injective $R$-module, and $A/H$ is $h$-reduced.

**Proof.** Let $H$ be the sum of all submodules of $A$ which are homomorphic images of injective $R$-modules. It is clear that $H$ is the unique largest submodule of $A$ which is a homomorphic image of an injective $R$-module. Suppose that $B/H$ is a homomorphic image of an injective $R$-module, where $B$ is a submodule of $A$ containing $H$. Then by Proposition 2.2 $B$ is a homomorphic image of an injective $R$-module. Therefore, $B = H$ and $B/H = 0$.

**Proposition 2.4.** Let $R$ be an integral domain with quotient field $Q$ such that $\text{hd}_R Q = 1$. Let $D$ be a divisible module over $R$, and let $H$ be a submodule of $D_T$ such that $H$ is a homomorphic image of an injective $R$-module. Then $D_T$ is a direct summand of $D$ if and only if $D_T/H$ is a direct summand of $D/H$.

**Proof.** Suppose that $D_T$ is a direct summand of $D$, and let $S$ be a complementary summand of $D_T$ in $D$. Then $D/H \cong D_T/H \oplus S$, and since $D_T/H$ is the torsion submodule of $D/H$, $D_T/H$ is a direct summand of $D/H$. Conversely, suppose that $D/H = D_T/H \oplus G/H$, where $G$ is a submodule of $D$ containing $H$. Now $G/H$ is torsion-free and divisible, hence injective. Thus by Proposition 2.2 and Theorem 1.1 $H$ is a direct summand of $G$. Let $L$ be a complementary summand of $H$ in $G$. Then it is clear that $D = D_T \oplus L$.

**Proposition 2.5.** Let $R$ be an integral domain with quotient field $Q$, and let $T$ be an $h$-reduced torsion $R$-module. Then $\text{Ext}^1_B(Q, T) = 0$ if and only if $T \cong \text{Ext}^1_B(Q/R, T)$.

**Proof.** Since $\text{Hom}_R(Q, T) = 0$, we have an exact sequence:

$$0 \rightarrow \text{Hom}_R(R, T) \rightarrow \text{Ext}^1_B(Q/R, T) \rightarrow \text{Ext}^1_B(Q, T) \rightarrow 0.$$ 

It follows that if $\text{Ext}^1_B(Q, T) = 0$, then $T \cong \text{Ext}^1_B(Q/R, T)$. Conversely, if $T \cong \text{Ext}^1_B(Q/R, T)$, then the above exact sequence shows that $\text{Ext}^1_B(Q, T)$ is a torsion module. However, $\text{Ext}^1_B(Q, T)$ is torsion-free, and thus $\text{Ext}^1_B(Q, T) = 0$.

**Corollary 2.6.** Let $R$ be an integral domain with quotient field $Q$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Then the torsion submodule of a divisible $R$-module is always a direct summand if and only if the following two conditions hold:

1. $\text{hd}_R Q = 1$.
2. $T \cong \text{Ext}^1_R(Q/R, T)$, whenever $T$ is an $h$-reduced, torsion, divisible $R$-module.

**Proof.** The necessity follows from Theorem 1.2 and Proposition 2.5; the sufficiency follows from Corollary 2.3 and Propositions 2.4 and 2.5.

**Proposition 2.7.** Let $R$ be an integral domain with quotient field $Q$ such that $\text{hd}_R Q = 1$ and $\text{gl. dim. } R \leq 2$. Let $S$ be any torsion-free $R$-module. Then $\text{hd}_R S \leq 1$. Thus if $A$ is an $R$-module such that $A_T$ is a homomorphic image of an injective $R$-module, then $A_T$ is a direct summand of $A$.

**Proof.** Let $B$ be any $R$-module. Then from the exact sequence:

$$0 \to S \to Q \otimes_R S \to Q/R \otimes_R S \to 0$$

we derive the exact sequence:

$$\text{Ext}^2_R(Q \otimes_R S, B) \to \text{Ext}^2_R(S, B) \to \text{Ext}^3_R(Q/R \otimes_R S, B).$$

Since $\text{hd}_R Q = 1$ and $\text{gl. dim. } R \leq 2$, the two end modules are zero. Thus $\text{hd}_R S \leq 1$, and the rest of the theorem follows immediately.

2. **Krull dimension = 1.**

Throughout this section $R$ will be a Noetherian integral domain with the property that nonzero prime ideals are maximal. We will let $Q$ be the quotient field of $R$ and $K = Q/R$.

**Definition.** Let $A$ be an $R$-module and $M$ a prime ideal of $R$. We will say that $A$ is $M$-primary, if for any $x \not= 0 \in A$, the order ideal of $x$ is an $M$-primary ideal. If $B$ is any $R$-module, and $A$ is the set of all elements of $B$ whose order ideal is $M$-primary (together with the element 0), then $A$ is an $M$-primary $R$-module which we will call the $M$-primary component of $B$.

**Lemma 3.1.** Let $B$ be any torsion $R$-module. Then $B$ is the direct sum of its $M$-primary components, $M$ ranging over the prime ideals of $R$. Furthermore, $B \otimes_R R_M$ is the $M$-primary component of $B$.

**Proof.** Let $\{M_a\}$ be the collection of nonzero prime ideals of $R$. By [3, Theorem 3.3] $E(B) = \sum_a \oplus E_a$, where $E_a$ is the $M_a$-component of $E(B)$. Let $B_a = B \cap E_a$; then $B_a$ is the $M_a$-component of $B$. Let $x \not= 0 \in B$; then $x = x_1 + \cdots + x_n$, where $x_i \in E_i$. Now $\cap_{i=2}^n M_i \subset M_1$.
hence there exists $s \in \cap_{i=1}^n M_i$ such that $s \in M_1$. Then there exists an integer $k > 0$ such that $s^k x_i = 0$ for $i = 2, \ldots, n$. Hence $s^k x = s^k x_1$.

There are elements $m \in M$ and $t \in R$ such that $1 = m + ts^k$. There is an integer $q > 0$ such that $m x_1 = 0$. Since $1 = m^q + rs^k$, $r \in R$, we have $x_1 = rs^k x_1 = rs^k x \in B_1$. Similarly $x_i \in B_i$ for $i = 2, \ldots, n$. Thus $B = \sum a \oplus B_a$.

Let $M_*$ be a prime ideal of $R$. Clearly $B_a \otimes_R R_{M_*} = 0$, if $M_a \neq M_*$. Thus $B \otimes_R R_{M_*} = B_* \otimes_R R_{M_*}$. It is easily seen that the canonical map $B_* \to B_v \otimes_R R_{M_*}$ is an epimorphism. However, since $B_*$ is $M_*$-primary, the kernel of this map is zero. Thus $B_* = B_v \otimes_R R_{M_*}$, and so $B \otimes_R R_{M_*} = B_*$. 

**Lemma 3.2.** $\text{hd}_R Q = 1$.

**Proof.** It is sufficient to prove that $\text{hd}_R K = 1$. By Lemma 3.1 $K = \sum a \oplus K_{M_a}$; thus it is sufficient to prove that $\text{hd}_R K_{M_*} = 1$. For this it is sufficient to prove that if $D$ is any divisible $R$-module, then $\text{Ext}_R^1(K_{M_*}, D) = 0$. Let $A$ be any extension of $D$ by $K_{M_*}$; then $A$ is a divisible $R$-module. We have $D = \sum a \oplus D_{M_*}$ and $A = \sum a \oplus A_{M_*}$. Clearly $D_{M_*} = D \cap A_{M_*}$. Hence for $a \neq v$, we have $D_{M_*} = A_{M_*}$. Thus we have an exact sequence:

$$0 \to D_{M_*} \to A_{M_*} \to K_{M_*} \to 0,$$

and all of the modules and mappings of this sequence are $R_{M_*}$-modules and mappings. Thus we can assume that $R$ is a local ring with a single nonzero prime ideal $M$.

Take $s \neq 0 \in M$, and let $S$ be the multiplicatively closed set consisting of the powers of $s$. Now $R_S \subseteq Q$; on the other hand, the prime ideals of $R_S$ and the prime ideals of $R$ not meeting $S$ are in 1-1 correspondence. Thus $R_S$ is a field, and $R_S = Q$. Therefore, $Q$ is a countably generated $R$-module; and thus $\text{hd}_R Q = 1$ by Theorem 1.3, or the remark following it.

**Theorem 3.3.** Every divisible $R$-module $D$ is a homomorphic image of an injective $R$-module; and thus $D_T$ is a direct summand of $D$.

**Proof.** Since $\text{hd}_R Q = 1$ by Lemma 3.2, it follows from Proposition 2.2 that we only need to prove that $D_T$ is a homomorphic image of an injective $R$-module. By Lemma 3.1 $D_T = \sum a \oplus D_{T_a}$, where $D_{T_a} = D_T \otimes_R R_{M_a}$ is a divisible $R_{M_a}$-module. As we have seen in Lemma 3.2, $Q$ is a countably generated $R_{M_*}$-module; and thus by Theorem 1.3, $D_{T_a}$ is a homomorphic image of an injective $R_{M_*}$-module. Thus $D_{T_a}$ is a homomorphic image of an injective $R$-module; and therefore, the same is true of $D_T$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
A CHARACTERIZATION OF ALGEBRAIC NUMBER FIELDS WITH CLASS NUMBER TWO

L. CARLITZ

Let \( Z = \mathbb{R}(\theta) \) denote an algebraic number field over the rationals with class number \( h \). It is familiar that \( h = 1 \) if and only if unique factorization into prime holds for the integers of \( Z \). For fields with \( h \leq 2 \) we have the following criterion.

**Theorem.** The algebraic number field \( Z \) has class number \( h = 2 \) if and only if for every nonzero integer \( \alpha \in Z \) the number of primes \( \pi_j \) in every factorization

\[
\alpha = \pi_1 \pi_2 \cdots \pi_k
\]

depends only on \( \alpha \).

Suppose first that \( h = 2 \) and consider the factorization into prime ideals

\[
\alpha = p_1 \cdots p_s r_1 \cdots r_t,
\]

where the \( p_j \) are principal ideals while the \( r_j \) are not. Then

\[
p_i = (\pi_i) \quad (j = 1, \ldots, s).
\]

Since \( h = 2 \), it follows that

\[
r_i r_j = (\rho_{ij}) \quad (i, j = 1, \ldots, t);
\]

moreover \( t \) must be even, \( = 2u \), say. Thus every factorization into primes implied by (2), for example

---

Received by the editors August 3, 1959.

1 Research sponsored by National Science Foundation grant NSF G-9425.