THE CONVERSE OF THE INDIVIDUAL ERGODIC THEOREM

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Let $(X, \mathcal{S}, m)$ be a measure space and let T be a measurable transformation of X into itself. The transformation T is called measure-preserving if $m(T^{-1}E)=m(E)$ for each measurable set E. The Individual Ergodic Theorem [1; 7] asserts that if T is measure-preserving and if f is an integrable function, then the averages

$$(1) \qquad \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

converge almost everywhere to a finite limit $f^*(x)$. It then follows that the limit function f^* is integrable and that $f^*(Tx)=f^*(x)$ almost everywhere.

This result can be applied to certain cases in which the given measure m is not preserved by the transformation T. In order to discuss this application, we recall some terminology for measures and transformations. If $(X, \mathcal{S})$ is a measurable space, and if p and m are two measures on $\mathcal{S}$, we shall say that p is absolutely continuous with respect to m, and shall write $p \ll m$, if $m(E)=0$ implies $p(E)=0$. The measures p and m are said to be equivalent if both $p \ll m$ and $m \ll p$, in which case we write $p \equiv m$.

If T is a measurable transformation on a measure space $(X, \mathcal{S}, m)$, then T defines a second measure on $\mathcal{S}$, denoted by mT^{-1} and defined by $mT^{-1}(E)=m(T^{-1}E)$. To say that T is measure-preserving is to say that $m=mT^{-1}$. A transformation T will be called an absolutely continuous transformation if $mT^{-1} \ll m$, and will be called nonsingular if $mT^{-1} \equiv m$. Nonsingular transformations are sometimes called measurability preserving transformations.

A measurable transformation T is called incompressible if $m(E-T^{-1}E)=0$ implies $m(T^{-1}E-E)=0$. In suggestive language, T is incompressible if and only if $E \subset T^{-1}E$ a.e. implies $E=T^{-1}E$ a.e. It is clear that T is incompressible if and only if $m(T^{-1}E-E)=0$ implies $m(E-T^{-1}E)=0$, and that an incompressible transformation is always nonsingular. A measure-preserving transformation on a finite measure space is incompressible, but this need not be the case

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on an infinite measure space. Furthermore, let T be a measurable transformation on a measurable space $(X, \mathcal{S})$ and let m and p be two equivalent measures on $\mathcal{S}$; it is clear that T is incompressible on $(X, \mathcal{S}, m)$ if and only if it is incompressible on $(X, \mathcal{S}, p)$. In particular, if T is a measurable transformation on a measure space $(X, \mathcal{S}, m)$, and if there exists an equivalent finite measure p on $\mathcal{S}$ which is invariant (that is, $p = pT^{-1}$), then T is incompressible on $(X, \mathcal{S}, m)$.

It is now simple to state a modification of the Individual Ergodic Theorem which holds if there is an equivalent finite invariant measure. Let f_E be the characteristic function of a measurable set. Since the invariant measure p is finite, this is an integrable function with respect to p, and the averages (1) converge almost everywhere $[p]$. Since $p \equiv m$, where m is the given measure, these averages converge almost everywhere $[m]$ also.

To what extent is the converse of the Individual Ergodic Theorem true? In other words, if the averages (1) converge almost everywhere, what can be said about the invariance of m? Following the discussion above, if T is a measure-preserving transformation on a measure-space $(X, \mathcal{S}, p)$ and if m is a measure equivalent to p and having exactly the same integrable functions, then the Individual Ergodic Theorem holds for T as a transformation on $(X, \mathcal{S}, m)$, even though m is not invariant. We may ask if this is the most general situation. In other words, if the averages (1) converge almost everywhere, is there an equivalent invariant measure?

A partial answer has been given by Y. N. Dowker [2]. Call a measurable transformation T invertible if T is one-one and if the inverse T^{-1} is also measurable. Dowker has shown that if T is an invertible transformation on $(X, \mathcal{S}, m)$ and if the averages (1) converge almost everywhere to a finite limit, then there exists a finite measure p on $\mathcal{S}$ such that $p = pT^{-1}$ and such that $mT^{-n} \ll p$ for all integers n, provided the measure m is itself finite. Moreover, if T is absolutely continuous, there is an invariant measure p such that $p \equiv m$.

The purpose of the present note is to present a version of the converse of the Individual Ergodic Theorem which does not rely on the invertibility of T. We shall show that the assumption of absolute continuity of T insures the existence of a nontrivial invariant finite measure $p \ll m$ (Theorem 1). If T is incompressible, we can assert that $p \equiv m$ (Theorem 3). The same technique also enables us to recapture Dowker's result for invertible absolutely continuous transformations (Theorem 2). The given measure space must be a finite measure space.

THEOREM 1. *Let* $(X, \mathcal{S}, m)$ *be a finite measure space and let* T *be a measurable transformation on* X *such that, for any measurable set* E, *the averages*

$$\frac{1}{n}\sum_{j=0}^{n-1} f_E(T^j x) \tag{2}$$

converge almost everywhere to finite limit $f_E^*(x)$, *where* f_E *is the characteristic function of* E. *If* T *is an absolutely continuous transformation, then there exists a finite measure* p *on* S *such that* $p=pT^{-1}$, $p\ll m$, *and such that* $p(E)=m(E)$ *if* E *is any invariant set.*

PROOF. Throughout the proof, let $f_n(x)$ denote the nth average in (2), for a measurable set E. We have $0\leqq f_n(x)\leqq 1$ for each n and x. The Bounded Convergence Theorem of Lebesgue asserts that the limit function f_E^* is integrable, and that the integrals $\int f_n(x)dm(x)$ converge to $\int f_E^*(x)dm(x)$. If we set

$$p(E) = \int f_E^*(x)dm(x), \tag{3}$$

then we have

$$p(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} m(T^{-j}E). \tag{4}$$

Note that the finiteness of m is used to assure the integrability of the constant function 1 in applying the Bounded Convergence Theorem.

Let $q_n(E)$ denote the nth average appearing in (4), so that $p(E)=\lim_{n\to\infty} q_n(E)$. Each q_n is clearly a finite measure on $\mathcal{S}$, and it follows at once that p is a non-negative and finitely additive function on $\mathcal{S}$. Since T is absolutely continuous, each $q_n\ll m$. By a standard result in measure theory [**3**, p. 159], p is countably additive, and $p\ll m$.

By an invariant set is meant a set E such that $m(E+T^{-1}E)=0$, where for any sets A, B, $A+B$ denotes the symmetric difference $(A-B)\cup(B-A)$. If E is invariant, then $m(E)=m(T^{-1}E)$. Since T is absolutely continuous, a simple induction shows that $T^{-j}E$ is invariant for each non-negative integer j, and it follows that $m(E)=m(T^{-j}E)$. Formula (4) shows at once that m and p agree on invariant sets. In particular, since $X=T^{-1}X$, then $m(X)=p(X)$, so that p is finite.

By hypothesis, there is a set A with $m(A)=0$ such that $f_E^*(x)=\lim f_n(x)$ if x is not in A. If x is not in $T^{-1}A$, then $f_E^*(Tx)=\lim f_n(Tx)$, and the absolute continuity of T implies that $m(T^{-1}A)=0$. Since $f_n(Tx)-f_n(x)=n^{-1}(f_E(T^nx)=f_E(x))$, it follows that for x not in

$A \cup T^{-1}A$, $f_E^*(Tx) = f_E^*(x)$. If B plays the same role for $f_{T^{-1}E}$ that A plays for f_E, it is manifest that $f_{T^{-1}E}^*(x) = f_E^*(Tx)$ for x not in $B \cup T^{-1}A$. Consequently $f_{T^{-1}E}^*(x) = f_E^*(x)$ almost everywhere $[m]$. Integrating with respect to m yields $p(T^{-1}E) = p(E)$, and completes the proof.

We remark that the equality of p and m on invariant sets rules out any possibility that p is a trivial invariant measure.

THEOREM 2 (Y. N. DOWKER). *Let T be a measurable transformation on a finite measure space $(X, \mathcal{S}, m)$ such that the averages* (2) *converge almost everywhere to a finite limit for any measurable set E. If T is absolutely continuous and invertible, then there exists a finite measure p on $\mathcal{S}$ such that $p = pT^{-1}$, $p \equiv m$, and such that p and m agree on invariant sets.*

PROOF. We need only prove that the measure p of Theorem 1 has the property that $m \ll p$. Since we have $p \ll m$, the Radon-Nikodym Theorem enables us to write $p(E) = \int_E w(x)dm(x)$ for each measurable set E, where w is a measurable function. Since $p(E)$ is non-negative for each E, then $0 \leqq w(x)$ almost everywhere $[m]$. To show that $m \ll p$ is equivalent to showing that $0 < w(x)$ almost everywhere $[m]$. Set $N = \{x \in X: w(x) = 0\}$. Then $p(N) = 0$. If T is invertible, we have $p(TE) = p(E)$ as well as $p(T^{-1}E)$, so that $p(T^{-j}N) = 0$ for each integer j. Set $B = \bigcup_{j=-\infty}^{+\infty} T^{-j}N$, so that $p(B) = 0$. Since B is an invariant set, $m(B) = p(B) = 0$, and hence $m(N) = 0$ also. Thus $0 < w(x)$ almost everywhere $[m]$, and the proof is complete.

The same argument can be applied in the case of an incompressible transformation.

THEOREM 3. *Let $(X, \mathcal{S}, m)$ be a finite measure space and let T be a measurable transformation on X such that the averages* (2) *converge almost everywhere to a finite limit, for each measurable set E. If T is incompressible, then there exists a finite measure p on $\mathcal{S}$ such that $p = pT^{-1}$, $p \equiv m$, and such that p and m agree on invariant sets.*

PROOF. If T is incompressible, it is absolutely continuous and Theorem 1 applies. Write $p(E) = \int_E w(x)dm(x)$, as in the proof of Theorem 2, and set $N = \{x \in X: w(x) = 0\}$, $B = \bigcup_{j=0}^{\infty} T^{-j}N$. As before $p(B) = 0$. Now $T^{-1}B \subset B$, so that $m(B - T^{-1}B) = 0$, because of the incompressibility of T. Since m is finite, $m(B) = m(T^{-1}B)$, and by induction, $m(B) = m(T^{-j}B)$ for each non-negative integer j. From (4), we obtain $m(B) = p(B) = 0$. Hence $m(N) = 0$ and therefore $m \equiv p$.

Theorem 3 is the converse of the modified Individual Ergodic

Theorem given above, provided the measure space is finite. Let us give an example which illustrates the difficulty when m is infinite.

Let $(X, \mathcal{S}, \alpha)$ be the real line under Lebesgue measure α. A measurable transformation T on this (or any) measure space is called ergodic if, for any invariant set E, either $\alpha(E)=0$ or $\alpha(X-E)=0$. If T is ergodic, invertible, and measure-preserving on $(X, \mathcal{S}, \alpha)$, then T is incompressible, but there is no equivalent invariant *finite* measure [5, p. 84]. However, since T is measure-preserving on $(X, \mathcal{S}, \alpha)$, the averages (1) converge almost everywhere $[\alpha]$ for any integrable function. In the case of a finite space, the convergence of (2) is a weaker assumption than the convergence of (1), at least formally. The example shows that the convergence of (1) and incompressibility will not, for infinite spaces, produce a *finite* invariant measure.

Suppose, in the above example, α is replaced by an equivalent finite measure m. The transformation T is still ergodic, incompressible and invertible on $(X, \mathcal{S}, m)$, and there is still no equivalent finite invariant measure. It follows from Theorem 3 that there is a measurable set E such that the averages do not converge almost everywhere to a finite limit. Moreover, since the averages in (2) are non-negative and bounded by 1, it cannot be asserted that they converge almost everywhere either to a finite limit or to $+\infty$. Such an exceptional set necessarily has infinite Lebesgue measure. Since these negative assertions apply to any ergodic measure-preserving T on the real line, they show that extensions of the Individual Ergodic Theorem to classes of functions which are not (in some sense) integrable are generally impossible. In particular, the extension indicated by Y. N. Dowker [2, Theorem 1′] is limited to finite measure spaces.

This example can be put to another use. Consider the transformation T on the finite measure space $(X, \mathcal{S}, m)$ as indicated above. Since T is incompressible, it is nonsingular, and we can write $m(T^{-j}E) = \int_E w_j(x)dm(x)$, where w_j is a measurable function with $0 < w_j(x)$ almost everywhere. The General Ergodic Theorem of Halmos [4] asserts that the weighted averages

$$(5) \qquad \frac{\sum_{j=0}^{n-1} f(T^j x) w_j(x)}{\sum_{j=0}^{n-1} w_j(x)}$$

converge almost everywhere to a finite limit, for any integrable f. (This is actually the special case of the General Ergodic Theorem established earlier by Hurewicz [6].) If we set $f=f_E$, where E is one of the exceptional sets mentioned above, then f is an integrable function for which the conclusion of the General Ergodic Theorem is valid, but for which the conclusion of the Individual Ergodic Theorem does not hold. This means that the General Ergodic Theorem, and the theorem of Hurewicz in particular, are nontrivial extensions of the Individual Ergodic Theorem. In the case of a finite measure space, this means that the convergence of (5) may hold when (1) does not converge, for an integrable f. In the case of an infinite measure space, this means that there are measurable functions f for which (5) converges while (1) does not converge, even for a measure preserving T, provided the weighting functions w_j are suitably chosen.

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