It is of course well known that \( 1 - 1 + 1 - 1 + \cdots \) is Abel-summable to \( 1/2 \), that is to say
\[
1 - t + t^2 - t^3 + \cdots \to \frac{1}{2} \quad \text{as} \ t \to 1^-.
\]

It can also be shown that
\[
1 - t + t^4 - \cdots \pm t^n \cdots \to \frac{1}{2} \quad \text{as} \ t \to 1^-.
\]

If we consider, however, the rate at which these two functions approach their limit \((1/2)\) then we find that these are worlds apart! In fact
\[
(1 - t + t^2 \cdots) - \frac{1}{2} = \frac{1}{1 + t} - \frac{1}{2} = \frac{1 - t}{2(1 + t)} \sim \frac{1 - t}{4},
\]
whereas an application of the functional equation for the \(\theta\)-function gives
\[
(1 - t + t^4 \cdots) - \frac{1}{2} \sim \left(\frac{\pi e^{-\pi}}{1 - t}\right)^{1/2} \exp\left(-\frac{\pi^2}{4} \cdot \frac{1}{1 - t}\right).
\]

In this note we show that this anomalous behavior persists for the other functions
\[
(1) \quad f_k(t) = \sum_{n=0}^{\infty} (-1)^n t^{nk}, \quad k = 1, 2, \cdots,
\]
that namely \(f_k(t)\) approaches \(1/2\) very slowly for \(k\) odd and very quickly for \(k\) even.

**Theorem.** If \(f_k(t)\) are defined by (1) then
A. for any fixed odd \(k\) there is a \(c > 0\) such that \(|f_k(t) - 1/2| > c(1 - t)\),
B. for any fixed even \(k\) there is a \(c > 0\) such that
\[
|f_k(t) - 1/2| < A \exp \left(-c/(1 - t^\alpha)\right), \quad \alpha = 1/(k - 1).
\]

**Proof.** Our principal tool will be the Mellin inversion formula which gives

Received by the editors December 11, 1958 and, in revised form, August 1, 1959.
From this we obtain, by a term by term integration,
\[
\frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \Gamma(s) x^{-s} ds = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k^2}} = 1 - f_k(e^{-x}).
\]

Now note that
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k^2}} = \left(1 - \frac{2}{2^{k^2}}\right) \sum_{n=1}^{\infty} \frac{1}{n^{k^2}} = (1 - 2^{1-k^2})\zeta(k)\zeta(s)
\]
and obtain
\[
\frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \Gamma(s) x^{-s}(1 - 2^{1-k^2})\zeta(k) ds = 1 - f_k(e^{-x}).
\]

We now shift the contour from the vertical line Re \( s = 2 \) to the line Re \( s = -1/k \). (It is assumed that \( k > 1 \).) The estimates
\[
|\Gamma(s)| < Ae^{-c|t|}, \quad |\zeta(k)| < |t|^A, \quad t = \text{Im } s, \quad A, c > 0,
\]
are known to be valid in \(-1/k \leq \text{Re } s \leq 2, |t| > 1\) and so the Residue theorem is applicable.

However, in \(-1/k \leq \text{Re } s \leq 2\), the only pole is seen to be at \( s = 0 \) with residue \((1 - 2)\zeta(0) = 1/2\) and so we obtain
\[
\frac{1}{2\pi i} \int_{-1/k-\infty}^{-1/k+\infty} \Gamma(s) x^{-s}(1 - 2^{1-k^2})\zeta(k) ds = \frac{1}{2} - f_k(e^{-x}).
\]

Let us now utilize the functional equation of the \( \zeta \)-function, namely
\[
\zeta(s) = \frac{1}{\pi} (2\pi)^s \sin \frac{\pi}{2} s \Gamma(1 - s)\zeta(1 - s).
\]

If we introduce this into (3) and then call \( 1 - ks = z \) the result is
\[
-\frac{1}{k\pi i} \int_{2-\infty}^{2+\infty} x^{-s} \Gamma\left(\frac{1 - s}{k}\right) \Gamma(z) \cos \frac{\pi}{2} x^{(1-1/k)}(1 - 2^{-s})\zeta(z) dz
\]
\[
= \frac{1}{2} - f_k(e^{-x}).
\]

Proof of A. Here \( k \) is odd. If we shift the contour from 2 to \( 2k \) then the residue theorem is again applicable and it gives one residue at \( z = k+1 \), namely \( cx \) where
\[ c = \frac{2(k)!}{\pi^{k+1}} \left( 1 - 2^{-k(k+1)/2}\delta \right)^k(1 + 1)(-1)^{(k-1)/2}. \]

The remaining contour integral is the same as that in (5) with the limits changed to \(2k-i\infty, 2k+i\infty\). This clearly has magnitude

\[ \leq x^{(2k-1)/k} \left| \int_{2k-i\infty}^{2k+i\infty} M \cdot \left| \Gamma\left( \frac{1-z}{k} \right) \right| \, dz \right| = cx^{(2k-1)/k} \]

and so finally, by (5)

\[ \frac{1}{2} - f_k(e^{-x}) = cx + O(x^{2-1/k}), \quad k > 1, \]

which yields A immediately.

**Proof of B.** Here \( k \) is even and so the integrand in (5) is analytic for all \( z, \text{Re} z > 0 \). We may then shift the contour to the right to the line \( \text{Re} z = kM + 1 \) \((M = \text{half an odd integer})\) and no residues will be introduced.

We now estimate the resulting integral. We have

\[ |\pi^{-x}x^{(x-1)/k}(1 - 2^{-x})\xi(z)| < x^M, \]

\[ |\Gamma(z) \cos \frac{\pi}{2} z| \leq (kM)!, \]

\[ |\Gamma\left( \frac{1-z}{k} \right)| = |\Gamma\left( -M + \frac{iy}{k} \right)|, \quad (y = \text{Im} z), \]

\[ \leq \left| \frac{\Gamma\left( +\frac{1}{2} - \frac{iy}{k} \right)}{\Gamma\left( M - \frac{1}{2} \right)!} \cdot \frac{1}{\left( M - \frac{1}{2} \right)!} \right| \]

\[ = \left( \frac{\pi}{\cosh \frac{y}{k}} \right)^{1/2} \cdot \frac{1}{\left( M - \frac{1}{2} \right)!} \]

and so the resulting integral is in magnitude

\[ \leq c_1 \frac{(kM)!}{\left( M - \frac{1}{2} \right)!} x^M \leq c_2 M^{(k-1)M} x^M. \]

If \( M \) is now chosen close to \( x^{-\alpha}/e, \alpha = 1/k - 1 \) then the resulting estimate becomes
and $B$ follows immediately. This completes the proof.

These results seem to point to the fact that of all the series $\sum (-1)^n f(n)$ the one which goes "fastest" to $1/2$ is $\sum (-1)^n n^3$. I wish to thank J. Korevaar for pointing out that this is actually the case, in the following sense:

We have seen that $|\sum (-1)^n n^3 - 1/2| \leq e^{-c/(1-\theta)}$. On the other hand, it is impossible that

$$\left| \sum (-1)^n f(n) - \frac{1}{2} \right| \leq e^{-\phi(t)/(1-t)}$$

where $\phi(t) \geq 0$ is unbounded as $t \to 1$. It is, in fact, a theorem of Korevaar [3], that if

$$\sum A_n e^{-nu} \leq w(u), \quad u > 0,$$

and

$$A_n \geq -1, \quad \liminf_{u \to 0^+} u \log w(u) = -\infty,$$

then $\sum A_n e^{-nu} = \text{constant}$.

Changing $u$ into $\log(1/t)$ immediately yields the result stated above.

**Bibliography**


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