A NOTE ON LACUNARY FOURIER SERIES

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A theorem of Kolmogoroff [2, p. 73] states that if the Fourier series \( \mathfrak{S}[f] \) of an \( L \)-integrable function \( f(x) \) has an infinity of gaps \( (n_r, n'_r) \) for which \( n'_r/n_r \geq \lambda > 1 \), (gaps of Hadamard's type), then \( s_{n_r} \to f \) for almost all \( x \), and it can be concluded that if \( f(x) \) is continuous then this is valid for all \( x \). From this theorem one can derive another theorem of Kolmogoroff: If \( f \in L^2 \) and \( n_{r+1}/n_r \geq \lambda > 1 \) then \( s_{n_r} \to f \) for almost all \( x \). Recently R. Gosselin [1] has proved a similar theorem with considerably larger subsequences \( (n_r, n'_r) \), although with less precision in locating the indices.

In this note, we prove the following theorem for gaps where \( n'_r/n_r \to 1 \).

**Theorem.** Let \( f(x) \) be continuous at \( x = x_0 \), and let \( \mathfrak{S}[f] \) be a lacunary Fourier series with an infinity of gaps \( (n_r, n'_r) \) for which \( n'_r - n_r \to \infty \), \( n'_r/n_r \to 1 \) and

\[
\omega\left(x_0, \frac{\pi}{n'_r - n_r}\right) \log \left(1 - \frac{n_r}{n'_r}\right) \to 0,
\]

with

\[
\omega(x_0, \delta) = \sup_{0 < |t| \leq \delta} \left\{ |f(x_0 + t) - f(x_0)| \right\};
\]

then

\( s_{n_r}(x_0) \to f(x_0) \).

It is well known [2, p. 45] that \( |f(x_0 + h) - f(x_0)| = o(\log 1/|h|)^{-1} \) does not ensure the convergence of \( \mathfrak{S}[f] \) at the point \( x_0 \). From the above theorem it follows that in this case \( \mathfrak{S}[f] \) converges if, for example, it is a lacunary Fourier series with \( n_r = \nu^{1+\epsilon}, n_{r+1} = (\nu + 1)^{1+\epsilon} \) for every \( \epsilon > 0 \).

**Proof of Theorem.** Take \( n_r = n, n'_r = m \) and denote by \( D_n(t) \) and \( K_n(t) \) the Dirichlet and Fejér kernels, i.e.

\[
D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt, \quad (n + 1)K_n(t) = \sum_{r=0}^{n} D_r(t) = \frac{\sin^2 (n + 1)t/2}{2 \sin^2 t/2}.
\]

From the identity

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\[ mK_{m-1}(t) - nK_{n-1}(t) = \sum_{r=n}^{m-1} D_r(t) \]
\[ = (m - n)D_n(t) + \sum_{r=1}^{m-n-1} (m - n - r) \cos(n + r)t, \]

we have in virtue of the lacunarity of \( \mathcal{E}[f] \)

\[ s_n - f(x_0) = \frac{1}{\pi} \int_0^\pi \phi(x_0, t)D_n(t) \]
\[ = \frac{1}{\pi(m - n)} \int_0^\pi \phi(x_0, t)[mK_{m-1}(t) - nK_{n-1}(t)]dt, \]

with

\[ \phi(x_0, t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0). \]

The last integral can be written in the form

\[ \frac{1}{2\pi(m - n)} \left\{ \int_0^{\pi/(m - n)} + \int_{\pi/(m - n)}^\pi + \int_0^\pi \right\} \phi(x_0, t) \frac{\sin^2 m \frac{t}{2} - \sin^2 n \frac{t}{2}}{\sin^2 \frac{t}{2}} dt \]

\[ = I_1 + I_2 + I_3 \]
say. We have for \( I_1 \)

(2) \[ |I_1| \leq \frac{2}{\pi(m - n)} \omega(x_0, \frac{\pi}{m - n}) \int_0^{\pi/(m - n)} |\sin^2 mt - \sin^2 nt| \frac{dt}{\sin^2 t}. \]

It remains to estimate

\[ I_1' = \int_0^{\pi/(2(m - n))} \frac{|\sin^2 mt - \sin^2 nt|}{\sin^2 t} dt. \]

Taking \( mt = \tau, n/m = \xi \) we obtain

(3) \[ I_1' \leq \frac{\pi^2}{4} m \int_0^{\pi/(2(m - n))} \frac{|\sin^2 \tau - \sin^2 \xi \tau|}{\tau^2} d\tau. \]

It follows from

\[ \sin^2 \tau - \sin^2 \xi \tau = \sin(1 - \xi)\tau \sin(1 + \xi)\tau \]

and from

\[ \sin^2 \tau \text{ and } \sin^2 \xi \tau \]
that the integral of the right-hand side of (2) is less than

$$\int_0^\infty |\sin (1 - \xi) t \sin (1 + \xi) t| \frac{dt}{r^2}$$

$$\leq 4(1 - \xi^2) \int_0^\infty \frac{dt}{(1 + (1 + \xi)t)(1 + (1 - \xi)t)}$$

$$= 2(1 - \xi^2) \frac{1}{\xi} \log \frac{1 + \xi}{1 - \xi}.$$

In virtue of (2), (3) and $\xi = n/m$ we have

$$|I_1| \leq \pi \frac{m + n}{n} \omega \left(x_0, \frac{\pi}{m - n}\right) \log \frac{m + n}{m - n}$$

and the last expression for $m/n \to 1$ is equivalent to the left-hand side of (1).

On the other hand, we obtain

$$|I_2| \leq \frac{2}{2\pi(m - n)} \pi^2 \omega(x_0, \delta) \frac{m - n}{\pi} = \omega(x_0, \delta),$$

and

$$|I_3| \leq \frac{C}{(m - n)\delta^2}$$

with an absolute constant $C$.

Given an $\epsilon$ we can choose $\delta$ so that $\omega(x_0, \delta) < \epsilon$ and we can suppose $(m - n)$ so large that $I_1 + I_2 + I_3 = o(1)$; this proves the theorem.

References


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