A NOTE ON CERTAIN POLYNOMIAL ALGEBRAS[1]

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1. **The algebras A_r.** We discuss in this note the properties of certain polynomial algebras over an integral domain k, where k has characteristic 2. Special cases of these algebras arise in Algebraic Topology: first, as the mod 2 cohomology algebras of the classifying spaces for real vector space bundles; and secondly, as a certain subalgebra (defined by C. T. C. Wall, see [**4**]) of the Thom algebra of nonoriented differentiable manifolds [**3**].

Let r denote either a positive integer or ∞. We set

$$A_r = k[x_1, \cdots, x_i, \cdots; y_1, \cdots, y_i, \cdots],$$

where $1 \leqq i \leqq r$, if r is an integer; $1 \leqq i < \infty$, if $r = \infty$. Define a derivation β on A_r by setting

$$(1) \qquad \beta x_i = y_i, \qquad \beta y_i = 0, \qquad \beta \mid k = 0;$$

$$(2) \qquad \beta(uv) = (\beta u)v + u(\beta v), \qquad u,v \in A_r.$$

Using the fact that k has characteristic 2, one may easily show

$$(3) \qquad \beta \circ \beta = 0.$$

We prove[2] a simple result relating the structure of A_r to β, and then apply this result to the examples mentioned above.

Denote by P the subalgebra of A_r spanned by all the monomials in x_i^2. Set

(4) *S = submodule of A_r spanned by all the elements* $p x_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$.

Here $p \in P$, $i_1 < \cdots < i_a$, $a \geqq 1$; $j_1 \leqq j_2 \cdots \leqq j_b$, $b \geqq 0$; and $i_1 \leqq j_1$, if $b > 0$.

THEOREM 1. *$A_r = P \oplus \beta S \oplus S$ (k-module direct sum), where Kernel $\beta = P \oplus \beta S$, Image $\beta = \beta S$.*

PROOF. Let T be the submodule of A_r spanned by the elements $p x_{i_1} \cdots x_{i_c} y_{j_1} \cdots y_{j_d}$, where $p \in P$, $i_1 < i_2 < \cdots < i_c$, $c \geqq 0$; $j_1 \leqq j_2 \leqq \cdots \leqq j_d$, $d \geqq 1$; and $j_1 < i_1$, if $c > 0$. Clearly, as a k-module, $A_r = P \oplus T \oplus S$. Hence, the splitting in Theorem 1 is obtained when we show

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LEMMA 1. $T\oplus S=\beta S\oplus S$.

We prove this by first defining a k-module homomorphism, λ, from S to T. It suffices to define λ on the generating elements $s=px_{i_1}\cdots x_{i_a}y_{j_1}\cdots y_{j_b}$ of S (see (4)). For this we set

$$\lambda(s) = px_{i_2}\cdots x_{i_a}y_{i_1}y_{j_1}\cdots y_{j_b};$$

it is easily checked from the definition that $\lambda(s)\in T$. Now let $t=px_{i_1}\cdots x_{i_c}y_{j_1}\cdots y_{j_d}$ be an arbitrary generator of T. Let $s=px_{j_1}x_{i_1}\cdots x_{i_c}y_{j_2}\cdots y_{j_d}$. Then, $s\in S$ and $\lambda(s)=t$. Thus we have shown

$$\lambda(S) = T. \tag{5}$$

The homomorphism λ is related to the derivation β in the following way: again let $s=px_{i_1}\cdots x_{i_a}y_{j_1}\cdots y_{j_b}$ be a generator of S. Then by (1), (2), and (3), we have $\beta p=0$, and

$$\beta s = \lambda s + \omega.$$

Here

$$\omega = \sum_{t=2}^{a} px_{i_1}\cdots \hat{x}_{i_t}\cdots x_{i_a}y_{i_t}y_{j_1}\cdots y_{j_b}$$

if $a\geqq 2$, and ^ means "omit." It follows from the definition that each term in ω belongs to S. Hence,

$$\beta s \equiv \lambda s \mod S. \tag{6}$$

Therefore, by (5),

$$\beta S \equiv \lambda S \equiv T \mod S.$$

Hence, $\beta S\oplus S=T\oplus S$, completing the proof of Lemma 1.

In order to complete the proof of Theorem 1 we are left with showing that Kernel $\beta=P\oplus\beta S$ (since this implies at once that Image $\beta=\beta S$). By (1), (2), and (3), it is clear that $P\oplus\beta S\subset$Kernel β. The proof of the theorem is thus complete when we show that β restricted to S is a monomorphism. But since $S\cap T=0$, and $\lambda(S)=T$, this will follow at once from (6) and the following lemma.

LEMMA 2. *λ is a monomorphism.*

Now a k-basis for the submodule S consists of the totality of classes $s=px_{i_1}\cdots x_{i_a}y_{j_1}\cdots y_{j_b}$, where p is a monomial in the elements x_i^2 and the integers $i_1,\cdots,j_b$ satisfy the conditions given after (4). Furthermore, each class λs is a k-basis element for T. Hence, the proof of Lemma 2 consists in showing that if s and s' are distinct basis elements of S, then λs and $\lambda s'$ are distinct basis elements of T. This follows fairly easily from the definitions of S, T, and λ; we leave the details to the reader. With Lemma 2 proved, the proof of Theorem 1 is then complete.

2. **Examples.** We give here some examples of algebras of type A_r. Denote by $\tilde{G}_n$ the set of oriented n-dimensional subspaces of R^∞ (R=real numbers), $n=1, 2, \cdots$. $\tilde{G}_n$ may be topologized so as to be a CW-complex and is then the base space of the classifying bundle, $\tilde{\gamma}^n$, for oriented n-plane bundles (see [1] for details). It is well known that

$$H^*(\tilde{G}_n; Z_2) = Z_2[\tilde{W}_2, \tilde{W}_3, \cdots, \tilde{W}_n],$$

where $\tilde{W}_i$ is the ith Stiefel-Whitney characteristic class of the bundle $\tilde{\gamma}^n$. Denote by β_2 the Bockstein coboundary associated with the exact sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0.$$

β_2 is a derivation (see (2)) and

$$\beta_2\tilde{W}_{2i} = \tilde{W}_{2i+1}, \qquad 1 \leqq i \leqq [n/2],\ n \text{ odd};$$

$$\beta_2\tilde{W}_{2i} = \tilde{W}_{2i+1}, \qquad \beta_2\tilde{W}_n = 0, \qquad 1 \leqq i < [n/2],\ n \text{ even}.$$

Therefore, for each positive integer q,

(7) *$H^*(\tilde{G}_{2q+1}; Z_2)$ is an algebra of type A_q, with respect to the field $k=Z_2$.*

Since we may write $H^*(\tilde{G}_{2q}; Z_2)=Z_2[\tilde{W}_{2q}][\tilde{W}_2, \cdots, \tilde{W}_{2q-1}]$, we also have:

(8) *$H^*(\tilde{G}_{2q}; Z_2)$ is an algebra of type A_{q-1}, with respect to the integral domain $k=Z_2[\tilde{W}_{2q}]$.*

In another paper [2] we use the following result, obtained from Theorem 1, to obtain a direct sum splitting of $H^*(\tilde{G}_n; Z)$.

COROLLARY 1. (a) *In $H^*(\tilde{G}_{2q+1}; Z_2)$ we have*

$$\text{Kernel } \beta_2 = Z_2[\tilde{W}_2^2, \cdots, \tilde{W}_{2q}^2] \oplus \text{Image } \beta_2.$$

(b) *In $H^*(\tilde{G}_{2q}; Z_2)$ we have*

$$\text{Kernel } \beta_2 = Z_2[\tilde{W}_2^2, \cdots, \tilde{W}_{2q-2}^2, \tilde{W}_{2q}] \oplus \text{Image } \beta_2.$$

For a second example consider the algebra, $\mathfrak{N}$, of cobordism classes of (compact, differentiable) nonoriented manifolds. This is defined by Thom in [3]. C. T. C. Wall has recently defined a certain polynomial subalgebra, $\mathfrak{W}$, of $\mathfrak{N}$ together with a derivation (boundary operator), ∂, defined on $\mathfrak{W}$. Wall shows that the pair $\mathfrak{W}$, ∂ is an algebra of type A_∞. The integral domain k in this case is the polynomial algebra $Z_2[\omega_1, \omega_2, \cdots]$, where each ω_i is the (nonoriented) cobordism class of the complex projective space $P_{2^i}(C)$. We do not give the details of this here as the example will be discussed more fully in a forthcoming paper. There Theorem 1 will be used to give generators and relations for the Thom algebra, Ω, of *oriented* manifolds (see [3]).

3. **The nonoriented case.** As remarked above the algebras A_r describe abstractly the cohomology algebra $H^*(\tilde{G}_n; Z_2)$. In a separate paper [2] we will need analogous results for the cohomology algebra $H^*(G_n; Z_2)$. (Here G_n is the complex whose points are n-dimensional nonoriented subspaces of R^∞.) To this end we define

$$B_q = A_{q-1} \otimes Z_2[u, v] \qquad (1 \leqq q < \infty), \tag{9}$$

where A_{q-1} is defined over the field $k = Z_2$. (We set $A_0 = Z_2[1]$). Thus, B_q may be regarded as a mod 2 polynomial algebra on generators $x_1, x_2, \cdots, x_{q-1}; y_1, y_2, \cdots, y_{q-1}; u, v$. We want to define a linear endomorphism β^* on B_q with properties analogous to those of β. Do this by first defining a derivation $\bar{\beta}$ on $Z_2[u, v]$ by

$$\bar{\beta}u = u^2, \qquad \bar{\beta}v = uv,$$

and then setting

$$\beta^*(a \otimes b) = \beta a \otimes b + a \otimes \bar{\beta} b,$$

for $a \in A_{q-1}$, $b \in Z_2[u, v]$. As a companion to Theorem 1 we then obtain

THEOREM 2. *Kernel* $\beta^* = P^* \oplus$ *Image* β^*, *where*

$$P^* = Z_2[x_1^2, \cdots, x_{q-1}^2, v^2].$$

The proof of this follows from a simple result about vector spaces. Suppose we are given mod 2 vector spaces V_1 and V_2 each with a linear endomorphism β_1, respectively β_2. Define an endomorphism β on $V_1 \otimes V_2$ by

$$\beta(v_1 \otimes v_2) = \beta v_1 \otimes v_2 + v_1 \otimes \beta v_2,$$

for $v_i \in V_i$. Suppose further that for $i = 1, 2$,

$$\text{Kernel } \beta_i = A_i \oplus \text{Image } \beta_i,$$

for some summand $A_i \subset V_i$. A simple calculation then gives:

LEMMA 3. *Kernel* $\beta = (A_1 \otimes A_2) \oplus$ *Image* β.

Now one easily verifies that

$$\text{Kernel } \bar{\beta} = Z_2[v^2] \oplus \text{Image } \bar{\beta}.$$

Thus, from Theorem 1 and Lemma 3, we obtain

$$\text{Kernel } \beta^* = (Z_2[x_1^2, \cdots, x_{q-1}^2] \otimes Z_2[v^2]) \oplus \text{Image } \beta^*.$$

Identifying $Z_2[x_1^2, \cdots, x_{q-1}^2] \otimes Z_2[v^2]$ with $Z_2[x_1^2, \cdots, x_{q-1}^2, v^2]$, we obtain Theorem 2.

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