A NOTE ON CERTAIN POLYNOMIAL ALGEBRAS

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1. The algebras $A_r$. We discuss in this note the properties of certain polynomial algebras over an integral domain $k$, where $k$ has characteristic 2. Special cases of these algebras arise in Algebraic Topology: first, as the mod 2 cohomology algebras of the classifying spaces for real vector space bundles; and secondly, as a certain subalgebra (defined by C. T. C. Wall, see [4]) of the Thom algebra of nonoriented differentiable manifolds [3].

Let $r$ denote either a positive integer or $\infty$. We set

$$A_r = k[x_1, \ldots, x_i, \ldots; y_1, \ldots, y_i, \ldots],$$

where $1 \leq i \leq r$, if $r$ is an integer; $1 \leq i < \infty$, if $r = \infty$. Define a derivation $\beta$ on $A_r$ by setting

(1) $\beta x_i = y_i, \quad \beta y_i = 0, \quad \beta k = 0;$

(2) $\beta(uv) = (\beta u)v + u(\beta v), \quad u, v \in A_r.$

Using the fact that $k$ has characteristic 2, one may easily show

(3) $\beta \circ \beta = 0.$

We prove a simple result relating the structure of $A_r$ to $\beta$, and then apply this result to the examples mentioned above.

Denote by $P$ the subalgebra of $A_r$ spanned by all the monomials in $x^2$. Set

(4) $S$ = submodule of $A_r$ spanned by all the elements $px_i x_{i_1} \ldots x_{i_a} y_{j_1} \ldots y_{j_b},$

where $p \in P, \quad i_1 < \cdots < i_a, \quad a \geq 1; \quad j_1 \leq j_2 \cdots \leq j_b, \quad b \geq 0; \quad$ and $i_1 \leq j_i, \quad$ if $b > 0.$

**Theorem 1.** $A_r = P \oplus \beta S \oplus S$ (k-module direct sum), where Kernel $\beta = P \oplus \beta S$, Image $\beta = \beta S.$

**Proof.** Let $T$ be the submodule of $A_r$ spanned by the elements $px_{i_1} \cdots x_{i_c} y_{j_1} \cdots y_{j_d},$ where $p \in P, \quad i_1 < i_2 < \cdots < i_c, \quad c \geq 0; \quad j_1 \leq j_2 \leq \cdots \leq j_d, \quad d \geq 1; \quad$ and $j_1 < i_1, \quad$ if $c > 0.$ Clearly, as a $k$-module, $A_r = P \oplus T \oplus S.$ Hence, the splitting in Theorem 1 is obtained when we show

Received by the editors August 21, 1959.

1 This research has been partly supported by U. S. Air Force Contract AF 49(638)-79.

2 I would like to thank the referee for his simplifications in the proof of Theorem 1.
Lemma 1. $T \oplus S = \beta S \oplus S$.

We prove this by first defining a $k$-module homomorphism, $\lambda$, from $S$ to $T$. It suffices to define $\lambda$ on the generating elements $s = p x_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$ of $S$ (see (4)). For this we set

$$\lambda(s) = p x_{i_2} \cdots x_{i_a} y_{j_1} y_{j_2} \cdots y_{j_b};$$

it is easily checked from the definition that $\lambda(s) \in T$. Now let $t = p x_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$ be an arbitrary generator of $T$. Let $s = p x_{i_1} x_{i_2} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$. Then, $s \in S$ and $\lambda(s) = t$. Thus we have shown

$$\lambda(S) = T.$$  

The homomorphism $\lambda$ is related to the derivation $\beta$ in the following way: again let $s = p x_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$ be a generator of $S$. Then by (1), (2), and (3), we have $\beta p = 0$, and

$$\beta s = \lambda s + \omega.$$  

Here

$$\omega = \sum_{a=2}^{a} p x_{i_1} \cdots x_{i_a} y_{j_1} y_{j_2} \cdots y_{j_b}$$  

if $a \geq 2$, and $\cdot$ means "omit." It follows from the definition that each term in $\omega$ belongs to $S$. Hence,

$$\beta s \equiv \lambda s \mod S.$$  

Therefore, by (5),

$$\beta S \equiv \lambda S \equiv T \mod S.$$  

Hence, $\beta S \oplus S = T \oplus S$, completing the proof of Lemma 1.

In order to complete the proof of Theorem 1 we are left with showing that Kernel $\beta = P \oplus \beta S$ (since this implies at once that Image $\beta = \beta S$). By (1), (2), and (3), it is clear that $P \oplus \beta S \subset$ Kernel $\beta$. The proof of the theorem is thus complete when we show that $\beta$ restricted to $S$ is a monomorphism. But since $S \cap T = 0$, and $\lambda(S) = T$, this will follow at once from (6) and the following lemma.

Lemma 2. $\lambda$ is a monomorphism.

Now a $k$-basis for the submodule $S$ consists of the totality of classes $s = p x_{i_1} \cdots x_{i_a} y_{j_1} \cdots y_{j_b}$, where $p$ is a monomial in the elements $x_i$ and the integers $i_1, \cdots, j_b$ satisfy the conditions given after (4). Furthermore, each class $\lambda s$ is a $k$-basis element for $T$. Hence, the proof of Lemma 2 consists in showing that if $s$ and $s'$ are distinct
basis elements of \( S \), then \( \lambda s \) and \( \lambda s' \) are distinct basis elements of \( T \). This follows fairly easily from the definitions of \( S \), \( T \), and \( \lambda \); we leave the details to the reader. With Lemma 2 proved, the proof of Theorem 1 is then complete.

2. Examples. We give here some examples of algebras of type \( A_r \). Denote by \( G_n \) the set of oriented \( n \)-dimensional subspaces of \( \mathbb{R}^\infty \) (\( \mathbb{R} = \) real numbers), \( n = 1, 2, \ldots \). \( G_n \) may be topologized so as to be a CW-complex and is then the base space of the classifying bundle, \( \tilde{\gamma}^n \), for oriented \( n \)-plane bundles (see [1] for details). It is well known that

\[
H^*(G_n; \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{W}_2, \tilde{W}_3, \ldots, \tilde{W}_n],
\]

where \( \tilde{W}_i \) is the \( i \)th Stiefel-Whitney characteristic class of the bundle \( \tilde{\gamma}^n \). Denote by \( \beta_2 \) the Bockstein coboundary associated with the exact sequence

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]

\( \beta_2 \) is a derivation (see (2)) and

\[
\begin{align*}
\beta_2\tilde{W}_{2i} &= \tilde{W}_{2i+1}, & 1 \leq i \leq \lfloor n/2 \rfloor, n \text{ odd;} \\
\beta_2\tilde{W}_{2i} &= \tilde{W}_{2i+1}, & \beta_2\tilde{W}_n = 0, & 1 \leq i < \lfloor n/2 \rfloor, n \text{ even.}
\end{align*}
\]

Therefore, for each positive integer \( q \),

(7) \( H^*(G_{2q+1}; \mathbb{Z}_2) \) is an algebra of type \( A_q \), with respect to the field \( k = \mathbb{Z}_2 \).

Since we may write \( H^*(G_{2q}; \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{W}_2, \ldots, \tilde{W}_{2q-1}, \tilde{W}_{2q}] \), we also have:

(8) \( H^*(G_{2q}; \mathbb{Z}_2) \) is an algebra of type \( A_{q-1} \), with respect to the integral domain \( k = \mathbb{Z}[\tilde{W}_{2q}] \).

In another paper [2] we use the following result, obtained from Theorem 1, to obtain a direct sum splitting of \( H^*(G_n; \mathbb{Z}) \).

Corollary 1. (a) In \( H^*(G_{2q+1}; \mathbb{Z}_2) \) we have

\[
\text{Kernel } \beta_2 = \mathbb{Z}_2[\tilde{W}_2^2, \ldots, \tilde{W}_{2q}^2] \oplus \text{Image } \beta_2.
\]

(b) In \( H^*(G_{2q}; \mathbb{Z}_2) \) we have

\[
\text{Kernel } \beta_2 = \mathbb{Z}_2[\tilde{W}_2^2, \ldots, \tilde{W}_{2q-2}^2, \tilde{W}_{2q}] \oplus \text{Image } \beta_2.
\]

For a second example consider the algebra, \( \mathcal{R} \), of cobordism classes of (compact, differentiable) nonoriented manifolds. This is defined by Thom in [3]. C. T. C. Wall has recently defined a certain polynomial subalgebra, \( \mathfrak{B} \), of \( \mathcal{R} \) together with a derivation (boundary
operator), $\partial$, defined on $\mathcal{B}$. Wall shows that the pair $\mathcal{B}$, $\partial$ is an algebra of type $A_\infty$. The integral domain $k$ in this case is the polynomial algebra $\mathbb{Z}_2[\omega_1, \omega_2, \cdots]$, where each $\omega_i$ is the (nonoriented) cobordism class of the complex projective space $P_{2i}(\mathbb{C})$. We do not give the details of this here as the example will be discussed more fully in a forthcoming paper. There Theorem 1 will be used to give generators and relations for the Thom algebra, $\Omega$, of oriented manifolds (see [3]).

3. The nonoriented case. As remarked above the algebras $A_r$ describe abstractly the cohomology algebra $H^*(\tilde{G}_n; \mathbb{Z}_2)$. In a separate paper [2] we will need analogous results for the cohomology algebra $H^*(G_n; \mathbb{Z}_2)$. (Here $G_n$ is the complex whose points are $n$-dimensional nonoriented subspaces of $\mathbb{R}^\infty$.) To this end we define

$$B_q = A_{q-1} \otimes \mathbb{Z}_2[u, v] \quad (1 \leq q < \infty),$$

where $A_{q-1}$ is defined over the field $k = \mathbb{Z}_2$. (We set $A_0 = \mathbb{Z}_2[1]$). Thus, $B_q$ may be regarded as a mod 2 polynomial algebra on generators $x_1, x_2, \cdots, x_{q-1}; y_1, y_2, \cdots, y_{q-1}; u, v$. We want to define a linear endomorphism $\beta^*$ on $B_q$ with properties analogous to those of $\beta$. Do this by first defining a derivation $\tilde{\beta}$ on $\mathbb{Z}_2[u, v]$ by

$$\tilde{\beta}u = u^2, \quad \tilde{\beta}v = uv,$$

and then setting

$$\beta^*(a \otimes b) = \beta a \otimes b + a \otimes \tilde{\beta}b,$$

for $a \in A_{q-1}, b \in \mathbb{Z}_2[u, v]$. As a companion to Theorem 1 we then obtain

**Theorem 2.** Kernel $\beta^* = P^* \oplus \text{Image } \beta^*$, where

$$P^* = \mathbb{Z}_2[x_1^2, \cdots, x_{q-1}^2, v^2].$$

The proof of this follows from a simple result about vector spaces. Suppose we are given mod 2 vector spaces $V_1$ and $V_2$ each with a linear endomorphism $\beta_1$, respectively $\beta_2$. Define an endomorphism $\beta$ on $V_1 \otimes V_2$ by

$$\beta(v_1 \otimes v_2) = \beta v_1 \otimes v_2 + v_1 \otimes \beta v_2,$$

for $v_i \in V_i$. Suppose further that for $i = 1, 2$,

$$\text{Kernel } \beta_i = A_i \oplus \text{Image } \beta_i,$$

for some summand $A_i \subset V_i$. A simple calculation then gives:
Lemma 3. Kernel $\beta = (A_1 \otimes A_2) \oplus \text{Image } \beta$.

Now one easily verifies that

$$\text{Kernel } \bar{\beta} = Z_2[v^2] \oplus \text{Image } \bar{\beta}.$$  

Thus, from Theorem 1 and Lemma 3, we obtain

$$\text{Kernel } \beta^* = (Z_2[x_1^2, \ldots, x_{q-1}^2] \otimes Z_2[v^2]) \oplus \text{Image } \beta^*.$$  

Identifying $Z_2[x_1^2, \ldots, x_{q-1}^2] \otimes Z_2[v^2]$ with $Z_2[x_1^2, \ldots, x_{q-1}^2, v^2]$, we obtain Theorem 2.

Bibliography


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