THE AREA OF A NONPARAMETRIC SURFACE

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1. Introduction. Consider a continuous real-valued function \( f \) on Euclidean \( n \)-space, \( E_n \), and the associated map

\[
\bar{f} : E_n \to E_{n+1}, \quad \bar{f}(x) = (x_1, \ldots, x_n, f(x)) \text{ for } x \in E_n.
\]

It will be proved that for each finitely, rectilinearly triangulable subset \( W \) of \( E_n \) the \( n \)-dimensional Lebesgue area of \( \bar{f}(W) \) equals the \( n \)-dimensional Hausdorff measure of \( f(W) \), and that if this measure is finite then \( f(W) \) is Hausdorff \( n \) rectifiable.

For the special case \( n = 2 \) these results were obtained in [F1].

The following notation will be used:

- \( \mathcal{L}_m \) = \( m \)-dimensional Lebesgue measure over \( E_m \).
- \( K(x, r) = E_m \cap \{ y : |y-x| < r \} \) for \( x \in E_m \), \( r > 0 \).
- \( \alpha(m) = \mathcal{L}_m[K(x, 1)] \) for \( x \in E_m \).
- \( C(x, r) = E_m \cap \{ y : |y-x| \leq r \} \) for \( x \in E_m \), \( r > 0 \).
- \( L_m = m \)-dimensional Lebesgue area.
- \( \gamma^k_m = k \)-dimensional integralgeometric measure over \( E_m \).
- \( \mathcal{H}_m^k = k \)-dimensional Hausdorff measure over \( E_m \).

For \( X \subset E_m \), \( \psi^k_m(X) = \) the infimum of the sums

\[
\sum_{S \in \mathcal{F}} \alpha(k)2^{-k} (\text{diam } S)^k
\]

corresponding to all countable coverings \( \mathcal{F} \) of \( X \).

The terminology adopted here is consistent with [F1] and [F3], where detailed information concerning the basic concepts may be found.


There is a number \( C_n \) such that

\[
\psi^{n-1}_n(X) \leq C_n\mathcal{H}^{n-1}(\text{Bdry } X)
\]

for every bounded open subset \( X \) of \( E_n \).

We include here a short variant of Gustin’s argument, partly because the following lemma is useful also for other purposes.

Lemma. If \( A \) and \( B \) are compact subsets of \( E_n \) such that \( A \cup B \) is a convex set with diameter \( \delta \), then

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\[
\frac{\mathcal{L}_n(A) \cdot \mathcal{L}_n(B)}{\delta^n} \leq \frac{3c_n^{n-1} (A \cap B)}{\delta^{n-1}}.
\]

**Proof of the Lemma.** Assume \( \delta = 1 \), let \( a \) and \( b \) be the characteristic functions of \( A \) and \( B \), and whenever \( 0 \neq z \in E_n \) let \( p_z \) be the orthogonal projection mapping \( E_n \) onto the subspace perpendicular to \( z \). By Fubini's theorem

\[
\mathcal{L}_n(A) \cdot \mathcal{L}_n(B) = \int_{E_n} \int_{E_n} a(x)b(y)d\mathcal{L}_n y d\mathcal{L}_n x
\]

\[
= \int_{E_n} \int_{E_n} a(x)b(x + z)d\mathcal{L}_n z d\mathcal{L}_n x
\]

\[
= \int_{|z| \leq 1} \mathcal{L}_n(\{x: x \in A \text{ and } x + z \in B\})d\mathcal{L}_n z
\]

\[
\leq \int_{|z| \leq 1} 3c_n^{n-1} [p_z(A \cap B)]d\mathcal{L}_n z \leq \alpha(n) 3c_n^{n-1} (A \cap B),
\]

because every segment joining \( x \in A \) to \( x + z \in B \) meets \( A \cap B \).

**Proof of the covering theorem.** Each \( x \in X \) is the center of a spherical ball \( C(x, r) \) with

\[
\frac{\mathcal{L}_n[C(x, r) \cap X]}{\alpha(n)r^n} = \frac{1}{2};
\]

in fact this ratio depends continuously on \( r > 0 \), equals 1 for small \( r \), and approaches 0 as \( r \) approaches \( \infty \). From \([M, \text{Theorem 3.5}]\) one obtains a sequence of such balls \( C(x_i, r_i) \) which are disjoint and for which

\[
X \subset \bigcup_{i=1}^{\infty} C(x_i, 5r_i).
\]

Applying the lemma with

\[
A = C(x_i, r_i) \cap \text{Clos } X \quad \text{and} \quad B = C(x_i, r_i) - X
\]

one finds that

\[
\left[ \frac{\alpha(n)^2}{2^{n+1}} \right] \leq \alpha(n) \frac{3c_n^{n-1} [C(x_i, r_i) \cap \text{Bdry } X]}{(2r_i)^{n-1}}
\]

for each \( i \), hence
3. Density ratios. Let \( Y = \text{range} \overline{f} \). It will be shown that
\[
\psi_{n+1}^n[Y \cap K(p, r)] \leq C_n 2^{n/2+1} \mathfrak{f}_{n+1}^n[Y \cap K(p, 5r)]
\]
whenever \( p \in E_n \) and \( r > 0 \). Obviously (see [F1, 10.3] or [F2, 4.1])
\[
\psi_{n+1}^n[Y \cap K(p, r)] \leq 2^{n/2+1} \frac{\alpha(n)}{\alpha(n - 1)} r \psi_{n+1}^n(\overline{f}^{-1}[K(p, r)]).
\]

Choose a finitely, rectilinearly triangulable set \( Q \) for which
\[
\overline{f}^{-1}[K(p, 4r)] \subset Q \subset \overline{f}^{-1}[K(p, 5r)],
\]
assume \( Q \) is nonempty, hence \( \overline{f}^{-1}[K(p, 5r)] - Q \) is nonempty, and infer from [T, 3.8, 3.10] that
\[
L_n(\overline{f} | Q) = \mathfrak{f}_{n+1}^n(\overline{f}(Q)) < \mathfrak{f}_{n+1}^n[Y \cap K(p, 5r)].
\]

Use [T, 3.8] and [F3, 6.18] to secure a continuously differentiable real-valued function \( g \), with the associated map
\[
\tilde{g} : E_n \to E_{n+1}, \quad \tilde{g}(x) = (x_1, \ldots, x_n, g(x)) \quad \text{for} \ x \in E_n,
\]
such that
\[
\overline{f}^{-1}[K(p, r)] \subset \tilde{g}^{-1}[K(p, 2r)],
\]
\[
\tilde{g}^{-1}[K(p, 3r)] \subset \overline{f}^{-1}[K(p, 4r)],
\]
\[
\mathfrak{f}_{n+1}^n(\tilde{g}(Q)) = L_n(\tilde{g} | Q) < \mathfrak{f}_{n+1}^n[Y \cap K(p, 5r)].
\]

For \( 2r < t < 3r \) the preceding covering theorem implies
\[
\psi_{n+1}^n(\overline{f}^{-1}[K(p, r)]) \leq \psi_{n+1}^n(\{ x : | \tilde{g}(x) - p | < t \})
\]
\[
\leq C_n \mathfrak{f}_{n+1}^n(\{ x : | \tilde{g}(x) - p | = t \})
\]
\[
\leq C_n \mathfrak{f}_{n+1}^n[\tilde{g}(Q) \cap \{ y : | y - p | = t \}].
\]

From the Eilenberg inequality ([E] or [F2, 3.2]) one obtains
\[
r \psi_{n+1}^n(\overline{f}^{-1}[K(p, r)]) \leq C_n \int_{2r}^{3r} \mathfrak{f}_{n+1}^n[\tilde{g}(Q) \cap \{ y : | y - p | = t \}] dt
\]
\[
\leq C_n \frac{2\alpha(n - 1)}{\alpha(n)} \mathfrak{f}_{n+1}^n[\tilde{g}(Q)],
\]
whence the initial assertion follows.
For \( \mathcal{C}_{n+1} \) almost all \( p \in Y \) it is known from [F1, 10.1] that
\[
\limsup_{r \to 0^+} \alpha(n)r^{-n} \psi_{n+1}^n[Y \cap K(p, r)] \geq 2^{-n},
\]
and one may now conclude that
\[
\limsup_{s \to 0^+} \alpha(n)s^{-n} \Gamma_{n+1}^n[Y \cap K(p, s)] \geq 2^{-n}C_n^{-1}2^{-3n/2-2}.
\]

4. Area, measure and rectifiability. The preceding result implies in conjunction with [F1, 3.1] that
\[
\mathcal{C}_{n+1}(S) \leq 5^nC_n^{3n/2+2} \mathcal{F}_{n+1}(S) \quad \text{for } S \subset Y;
\]
in case \( \mathcal{F}_{n+1}(S) < \infty \) it follows from the structure theorems [F1, 9.6, 9.7] that \( S \) is \( \mathcal{C}_{n+1} \) rectifiable and
\[
\mathcal{C}_{n+1}(S) = \mathcal{F}_{n+1}(S).
\]
Moreover [T, 3.8] shows that
\[
L_n(\bar{f} \mid W) = \mathcal{F}_{n+1}^n(\bar{f}(W))
\]
whenever \( W \) is a finitely, rectilinearly triangulable subset of \( E_n \).

References


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The proof of this lemma should be corrected as follows: On line 4 replace "\( \delta/5 \)" by "\( \delta \)." Replace lines 10 to 17 by "For each \( S \subset F \) we choose a point \( x(S) \in B \cap S \). Since \( \psi(X) = \phi(X) \) whenever \( X \subset E_n \) and \( \text{diam } X < \epsilon \), we have \( \phi(B) \leq \sum_{s \in F} \phi(B \cap S) = \sum_{s \in F} \psi(B \cap C[x(S), \text{diam } S]) \leq \sum_{s \in F} \lambda C[x(S), \text{diam } S] = 2^{k} \sum_{s \in F} \lambda C[x(S), \text{diam } S] \leq (2^{k})^{1/2} \phi(B)." A similar correction is required in the proof of [F1, 3.6].

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