A NOTE ON METRIC DENSITY OF SETS OF REAL NUMBERS

N. F. G. MARTIN

Casper Goffman has shown that the set of points at which the metric density of a set of real numbers exists but is not zero or one is a set of the first category. As a partial converse to this result he showed that for every $F_\sigma$ set of measure zero there exists a measurable set whose density exists at every point of the $F_\sigma$ and has the value $1/2$. In this note we extend the last named theorem of Goffman to the following:

**THEOREM.** Let $Z$ and $\gamma$ be given where $Z$ is an $F_\sigma$ set of measure zero and $\gamma$ is a real number such that $0 < \gamma < 1$. Then there exists a measurable set $S$ such that the metric density of $S$ exists at every point of $Z$ and has the value $\gamma$.

**PROOF.** We shall assume that $Z \subset (0, 1)$. Let $Z = \bigcup_{k=1}^{\infty} Z_k$ where $Z_k$ is closed and of measure zero for $k = 1, 2, \ldots$.

We shall define four sequences, $\{G_k\}$, $\{T_k\}$, $\{E_k\}$, and $\{F_k\}$ of sets, where $G_{k+1} \subset G_k$ and $T_{k+1} \subset T_k$, as follows:

Let $G_1 = (0, 1)$, and $G_k$ be an open set which contains $Z - \bigcup_{n=1}^{k-1} Z_n$. Define $T_k$ to be the set $G_k - Z_k$ and require $G_{k+1} \subset T_k$. Since $Z_k$ is closed, $T_k$ is open and consists of a countable number of disjoint open intervals $I_{kj} = (a_{kj}, b_{kj})$. Since $m(I_{kj}) < 1$, where $m$ denotes Lebesgue measure, there exists an integer $N_{kj}$ such that

$$\frac{1}{N_{kj} + 1} \leq \frac{1}{2} m(I_{kj}) \leq \frac{1}{N_{kj}}.$$

Let $\alpha_n = a_{kj} + (1/2)m(I_{kj}) = b_{kj} - (1/2)m(I_{kj}) = \beta_n$ for $n = N_{kj}$ and for $n \geq N_{kj} + 1$ let

$$\alpha_n = a_{kj} + \frac{1}{n}, \quad \beta_n = b_{kj} - \frac{1}{n},$$

$$A_n^1(k, j) = \{x: \alpha_n^{kj} \leq x < \alpha_n^{kj}\},$$

$$A_n^2(k, j) = \{x: \beta_n^{kj} \leq x < \beta_n^{kj}\}.$$

The sets $A_n^i(k, j), i = 1, 2; n \geq N_{kj} + 1$ are disjoint and

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\[ I_{kj} = \bigcup_{n=1}^{\infty} \left[ A^1_n(k, j) \ A^2_n(k, j) \right]. \]

For each of the sets \( A_{n}(k, j) \) let \( A_{n}^{i,kj} \) be any measurable set contained in \( A_{n}(k, j) \) for which \( m(A_{n}^{i,kj}) = \gamma m(A_{n}(k, j)) , \ i = 1, 2. \) Let \( A_{n}^{i,F_n,kj} \) be defined by \( A_{n}^{i,F_n,kj} = A_{n}^{i}(k, j) - A_{n}^{i,kj}. \) Then

\[ m(A_{n}^{i,F_n,kj}) = (1 - \gamma)m(A_{n}^{i}(k, j)). \]

Finally let

\[ E_{k} = \bigcup_{j=1}^{\infty} \bigcup_{n>N_{kj}} \left( A_{n}^{1,kj} \cup A_{n}^{2,kj} \right), \]

\[ F_{k} = \bigcup_{j=1}^{\infty} \bigcup_{n>N_{kj}} \left( A_{n}^{1,F_n,kj} \cup A_{n}^{2,F_n,kj} \right). \]

The sets \( E_{k} \) and \( F_{k} \) are disjoint and \( T_{k} = E_{k} \cup F_{k}. \)

Returning to the sets \( G_{k} \) restrict \( G_{k+1} \) so that

\[ m(G_{k+1} \cap A_{n}^{i,kj}) \leq \frac{1}{n^k} m(A_{n}^{i,kj}), \]

(1) \[ m(G_{k+1} \cap A_{n}^{i,F_n,kj}) \leq \frac{1}{n^k} m(A_{n}^{i,F_n,kj}) \]

for \( i = 1, 2; n \geq N_{kj} + 1; j = 1, 2, \ldots, \) which is possible since \( Z \) has measure zero.

The set \( S \) defined by

\[ S = \bigcup_{k=1}^{\infty} (E_{k} - G_{k+1}) \]

has density \( \gamma \) at every point of \( Z. \)

For, let \( s \) be any element in \( Z, \) and let \( h \) be the smallest positive integer such that \( s \in Z_{h} \subset G_{h}. \) Let \( I \) be an open interval containing \( s \) and contained in \( G_{h}. \) Restricting \( m(I) < 1/2 \) there exists an integer \( p > 1 \) such that

(2) \[ \frac{1}{p + 1} \leq m(I) < \frac{1}{p} . \]

Since \( s \in Z_{h}, s \in T_{h} \) and by (2) if an end point of \( I \) falls in \( A_{n}^{i}(h, j) \) then \( n \geq p \) and

\[ m(A_{n}^{i}(h, j)) = \frac{1}{n(n - 1)} \leq \frac{2}{p(p + 1)} . \]
The interval $I$ consists of the following:
1. A set $H$ composed of disjoint sets $A^i_n(h, j)$ where $i = 1, 2$ and $n \geq p$.
2. A set $J$ which consists of two open or half open intervals, possibly empty, at the ends of $I$ each of whose lengths does not exceed $2/p(p + 1)$ so that $m(J) \leq 4/p(p + 1)$.
3. A set $N = Z \cap I$ of measure zero.

We will show first that

\[(\gamma - 4/p)m(I) \leq m(H \cap E_h) \leq \gamma m(I),\]
\[ (1 - \gamma - 4/p)m(I) \leq m(H \cap F_h) \leq (1 - \gamma)m(I). \]

We have

\[m(H \cap E_h) = \sum m(A^i_n E_{hj}^i) = \gamma \sum m(A^i_n(h, j)) = \gamma m(H) \leq \gamma m(I),\]
where the summation is taken over all $n, j$ and $i$ for which $A^i_n(h, j) \subset I$.

Also, $m(H \cap F_h) = (1 - \gamma)m(H) \leq (1 - \gamma)m(I)$.

From the maximum measure of $J$ and inequality (2) we have $m(J) \leq (4/p)m(I)$. Thus since $m(I) = m(H) + m(J)$, $m(H) \geq (1 - 4/p)m(I)$. Therefore,

\[m(H \cap E_h) \geq (\gamma - 4/p)m(I)\]

and

\[m(H \cap F_h) \geq (1 - \gamma - 4/p)m(I).\]

Thus inequalities (3) are satisfied.

Next it will be shown that for all positive integers $q$

\[m(G_{h+q} \cap H \cap E_h) \leq m(H \cap F_h)/p^q,\]
\[m(G_{h+q} \cap H \cap F_h) \leq m(H \cap E_h)/p^q.\]

From the inequalities (1) and the fact that $n \geq p$,

\[m(G_{h+q} \cap H \cap E_h) = \sum m(G_{h+q} \cap A^i_n E_{hj}^i)\]
\[ \leq \sum m(A^i_n E_{hj}^i)/n^q \leq m(H \cap E_h)/p^q\]

where the summations are taken over all $n, j, i$ for which $A^i_n(h, j) \subset I$.

The second inequality in (4) is obtained in the same way. Since $G_{h+q} \supset E_{h+q}$, from (4) and (3) we obtain

\[m(E_{h+q} \cap H \cap E_h) \leq \gamma m(I)/p^q,\]
\[m(E_{h+q} \cap H \cap F_h) \leq (1 - \gamma)m(I)/p^q,\]
for \( q = 1, 2, \ldots \). Now
\[
I = (H \cap E_h) \cup (H \cap F_h) \cup N \cup J,
\]
so
\[
m \left( I \cap \bigcup_{k=h}^{\infty} E_k \right) \leq m(H \cap E_h) + \sum_{k=h+1}^{\infty} m(E_k \cap H \cap E_h) \\
+ \sum_{k=h+1}^{\infty} m(E_k \cap H \cap F_h) + m(J) \\
< \left( \gamma + \frac{5}{p} - 1 \right) m(I).
\]

Since \( I \supset (H \cap E_h) \cup (H \cap F_h) \) and \( E_h \cap F_h = \emptyset \),
\[
I \cap (E_h - G_{h+1}) \supset (H \cap E_h) - (G_{h+1} \cap H \cap E_h)
\]
and
\[
m(I \cap (E_h - G_{h+1})) > (\gamma - 5/p) m(I).
\]

However,
\[
I \cap (E_h - G_{h+1}) \subset I \cap S \subset I \cap \bigcup_{k=h}^{\infty} E_k
\]
so that application of inequalities (5) and (6) gives
\[
\gamma - \frac{5}{p} < \frac{m(I \cap S)}{m(I)} < \gamma + \frac{5}{p - 1}.
\]

Bibliography

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