ALMOST PERIODIC SOLUTIONS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS NEAR POINTS OF NONLINEAR FIRST APPROXIMATION

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1. Introduction. Sufficient conditions for the existence of almost periodic (a.p. for short) solutions of systems of the form

\[ \dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + r(x_1, \ldots, x_n, t), \quad i = 1, 2, \ldots, n, \]

are known provided the characteristic values of the matrix \( a = (a_{ij}) \) have nonzero real parts; cf., for example, [1]. Here the \( r_i \) are uniformly a.p. in \( t \) for \( (x_1, \ldots, x_n) \) in some region of Euclidean \( n \)-space. Amerio [3] has obtained such conditions for somewhat more specialized systems where some characteristic values of \( A \) may be pure imaginary; however, all must be distinct from zero.

In case \( n = 2 \), sufficient conditions for the existence, uniqueness, and asymptotic stability of a.p. solutions are known even if one of the characteristic values of \( A \) is zero. In fact, the system considered in [2] is

\[ \dot{x} = y - F(x), \]
\[ \dot{y} = -g(x) + p(t), \]

where \( F(0) = g(0) = 0 \), and the derivatives \( F'(x) \) and \( g'(x) \) are such that \( F'(x) > 0 \), and \( g'(x) \geq 0 \) where \( g'(x) = 0 \) only when \( x = 0 \). In this paper, a system of essentially this type is considered; we require now that \( g'(x) \leq 0 \) where \( g'(x) = 0 \) only when \( x = 0 \), and if \( g'(0) = 0 \), then \( F'(0) = 0 \). We prove an existence theorem for a unique a.p. solution which in case \( F'(x) > 0 \) for \( x \neq 0 \), has certain stability properties. We observe that this system is equivalent to the single second order equation:

\[ \ddot{x} + F'(x)\dot{x} + g(x) = p(t). \]

Throughout this paper, conventional topological definitions and notation is used; i.e., a region is an open connected plane set; the closure, union, intersection of sets are denoted respectively by \( \overline{R} \), \( R \cup S \), \( R \cap S \) where \( R \) and \( S \) are sets; \( p \in S \) means \( p \) is a member of \( S \), etc.

Received by the editors July 15, 1959.

1 Sponsored by the Office of Ordnance Research, U. S. Army.
2. **The existence theorem.** We consider the system

\begin{align*}
\dot{x} &= y - F(x), \\
\dot{y} &= g(x) + p(t);
\end{align*}

here (i) \( p(t) \) is a.p. and \( k = \max |p(t)| \);
(ii) \( g(0) = F(0) = 0 \);
(iii) there exist numbers \( a, b, a < 0 < b \), such that \( g(x) > k \) for \( x \geq b \), \( g(x) < -k \) for \( x \leq a \);
(iv) \( g'(x) \) and \( F'(x) \) are continuous, and \( g'(x) \geq 0 \) for \( a \leq x \leq b \).

We define

\[ \phi(x, \xi) = \begin{cases} 
(F(x + \xi) - F(x))/\xi, & \xi \neq 0, \\
F'(x), & \xi = 0;
\end{cases} \]

\[ h(x, \xi) = \begin{cases} 
(g(x + \xi) - g(x))/\xi, & \xi \neq 0, \\
g'(x), & \xi = 0;
\end{cases} \]

and

\[ D(x, \xi) = \phi^2(x, \xi) - 4h(x, \xi). \]

In terms of these definitions we state and prove the following

**Theorem.** If \( D(x, \xi) \leq 0 \) for \( a \leq x \leq b \), \( a \leq x + \xi \leq b \), and \( D(x, \xi) = 0 \) only if \( \xi = 0 \), then system (2) has a unique a.p. solution. If also \( \phi(x, \xi) > 0 \) for \( \xi \neq 0 \), then this a.p. solution is asymptotically stable as \( t \to +\infty \) with respect to a class \( \Sigma_1 \) of solutions of (2), and asymptotically stable as \( t \to -\infty \) with respect to a class \( \Sigma_2 \) of solutions of (2).

The proof of this theorem is based on a series of lemmas which are stated and proved after the following remarks.

**Remark 1.** \( g'(0) = P(0) = 0 \) is possible; hence, if (2) is considered as a special case of (1), the matrix \( A \) is in that case given by

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

the characteristic values of which are both zero. Hence, the system is markedly nonlinear at the origin.

**Remark 2.** If \( a \leq x_n \leq b \), \( a \leq x_n + \xi_n \leq b \) for \( n = 1, 2, \ldots \), then \( D(x_n, \xi_n) \to 0 \) as \( n \to \infty \) implies \( \xi_n \to 0 \) as \( n \to \infty \); this clearly follows from the hypotheses of the theorem, and continuity.

The following lemmas concern themselves with the system

\begin{align*}
\dot{x} &= y - F(x), \\
\dot{y} &= g(x) + p(t);
\end{align*}

\[ x = y - F(x), \]

\[ y = g(x) + p(t), \]
where \( e(t) \) is in the closure of the set of functions \( \{ p(t+h) \} \), \( |h| < \infty \), the closure being with respect to the uniform norm on \( |t| < \infty \). Clearly \( |e(t)| \leq k \) for all \( t \). Conditions on \( g(x) \) and \( F(x) \) as before (involving \( k \)) are presumed.

We consider the region \( R \) of the \((x, y)\) plane bounded by the lines \( y = mx + c, x = a, x = b; \) here \( m > 0 \) is fixed but arbitrary, and \( c > 0 \) is such that for \( a \leq x \leq b \) and \( n = 1, 2 \):

\[
\frac{g(x) + (-1)^{nk}}{m} > \frac{F(x)}{(-1)^{nc} + mx - F(x)},
\]

and

\[
|mx + (-1)^{nc}| > F(x).
\]

We denote by \( \Gamma \) the boundary of \( R \), and by \( \Gamma_a \) and \( \Gamma_b \) the subsets of \( \Gamma \) respectively defined by \( x = a, ma - c < y < F(a) \) and \( x = b, F(b) < y < mb + c \).

**Lemma 1.** Let \((x(t), y(t))\) be a solution of (3) such that \((x(t_0), y(t_0)) \in R\) and \((x(t_2), y(t_2)) \in \overline{R}\) for some \( t_2 > t_0 \). Then there exists a \( t_1, t_0 < t_1 < t_2\), such that \((x(t_1), y(t_1)) \in \Gamma_a \cup \Gamma_b\), and such that \((x(t), y(t)) \in R\) for \( t > t_1 \). Also, either \( x(t_2) < a \) or \( x(t_2) > b \), in the former case, \((x(t_1), y(t_1)) \in \Gamma_a\), in the latter case \((x(t_1), y(t_1)) \in \Gamma_b\).

**Proof.** From (vi) above, the graph of \( y = F(x) \) for \( a < x < b \) is in \( R \). It follows from (v) that the slope of \( \Gamma \) along the lines \( y = mx \pm c \) is greater than the slope of the trajectory of any solution of (3) at points on these parts of \( \Gamma \), i.e., trajectories of (3) cannot leave \( R \) there as \( t \) increases.

Since \( y > F(x) \) implies \( \dot{x} > 0 \) and \( y < F(x) \) implies \( \dot{x} < 0 \), it follows that trajectories of (3) above \( y = F(x) \) move to the right as \( t \) increases, while those below \( y = F(x) \) move to the left. Hence, trajectories cannot leave \( R \) on the part of \( \Gamma \) on \( x = a \) above \( y = F(x) \) and on \( x = b \) below \( y = F(x) \).

It remains only to consider the trajectories at points \((a, F(a))\) and \((b, F(b))\) of \( \Gamma \). In fact, if the trajectory of a solution \((x(t), y(t))\) of (3) is such that \( x(t_0) = b, y(t_0) = F(b) \), then \( \dot{x}(t_0) = 0 \) and \( \dot{y}(t_0) > 0 \). Hence this trajectory has a vertical tangent at \((b, F(b))\), is on the right side of this tangent for \( t \neq t_0 \), and moves up across the graph of \( y = F(x) \) there as \( t \) increases. It follows that any trajectory leaving \( R \) on \( \Gamma_b \) cannot return to \( R \) for \( t \) sufficiently large.

A similar discussion applies to the situation at points of the form \((a_1, F(a_1))\) where \( a_1 \leq a \), except that at such points, trajectories of (3)
cross down over \( y = F(x) \) as \( t \) increases; we omit the details, and the proof of the lemma is complete.

**Lemma 2.** There exists a solution \((x(t), y(t))\) of (3) such that 
\[ (x(t), y(t)) \in \mathbb{R} \text{ for all } t \geq 0. \]

**Proof.** Let \( p \) be a point of \((x, y)\) plane; then we will denote by 
\[ \mu(p, t) \] the point \((x(t), y(t))\) where \((x(t), y(t))\) is the solution of (3) such that 
\[ (x(0), y(0)) = p. \] We also denote by \( p_1 \) and \( p_2 \) the points of \( \Gamma \)
given by \((a, F(a))\) and \((b, mb+c)\) respectively and by \( \Gamma_1 \) the part of 
\( \Gamma \) between these points but above the curve \( y = F(x) \). Let the set of 
points on \( \Gamma_1 \) be ordered so that if \( p \) and \( q \) are distinct points on \( \Gamma_1 \), 
then \( p > q \) means that the distance along \( \Gamma_1 \) from \( p \) to \( p_2 \) is greater 
than the distance along \( \Gamma_1 \) from \( q \) to \( p_2 \).

Consider the set \( S_b \) of points on \( \Gamma_1 \) such that if \( p \in S_b \), then for each 
\( q \in \Gamma_1 \), \( q < p \), there exists a \( T = T(q) \geq 0 \) such that \( \mu(q, T) \in \Gamma_b \). This 
set is clearly nonempty and in fact contains an open interval of \( \Gamma_1 \) 
which has \( p_2 \) as its end point. Similarly, \( \Gamma_1 - S_b \) contains an interval 
containing \( p_1 \) as end point. Since \( S_b \) is bounded above, it has a least 
upper bound (l.u.b. for short), say \( q_1 \neq p_1 \). Clearly \( q_1 \neq p_2 \). We shall 
show that \( \mu(q_1, t) \in \mathbb{R} \) for all \( t > 0 \). For if not, then by Lemma 1 there 
exists a \( T_1 \geq 0 \) such that \( \mu(q_1, T_1) \in \Gamma_a \cup \Gamma_b \). Suppose \( \mu(q_1, T_1) \in \Gamma_a \), 
and let \( h > 0 \) be fixed but arbitrary. Then by Lemma 1 there exists a 
region \( T_a \) in the half plane \( x < a \) such that \( \mu(q_1, T_1 + h) \in \mathbb{R} \). By 
continuity of solutions of (3) as functions of their initial conditions, 
there exists a region \( S_1 \) such that \( q_1 \in S_1 \), and if \( p \in S_1 \cap \Gamma_1 \), there 
\[ \mu(p, T_1 + h) \in \mathbb{R}, \] and hence, by Lemma 1 there exists a \( \theta, 0 < \theta < 1, \) 
such that \( \mu(p, \theta(T_1 + h)) \in \Gamma_a \). Since \( S_1 \cap \Gamma_1 \) is an open interval of \( \Gamma_1 \) 
containing \( q_1 \), this contradicts the assumption that \( q_1 \) is a l.u.b. for \( S_b \).

The case \( m(2i, p_2) \notin \mathbb{R} \) for some \( p_2 \geq 0 \) is ruled out in the same way; 
we omit the details.

**Lemma 3.** There exists a solution \((x(t), y(t))\) of (3) such that 
\[ (x(t), y(t)) \in \mathbb{R} \text{ for all } t. \]

**Proof.** From Lemma 2, there exists a solution \((x_0(t), y_0(t))\) of (3) 
such that \((x(t), y(t)) \in \mathbb{R} \text{ for } t \geq 0. \) From a result of Amerio [3], there 
exists a solution \((x(t), y(t))\) of (3) such that \((x(t), y(t)) \in \mathbb{R} \text{ for all } t. \) 
This proves the lemma.

**Lemma 4.** If \((x(t), y(t)) \in \mathbb{R} \text{ for all } t, \) then this solution is unique.

**Proof.** Suppose there exists a solution \((x_1(t), y_1(t))\) of (3) such that 
\((x_1(t), y_1(t)) \in \mathbb{R} \text{ for all } t, \) and such that if \( \xi(t) = x_1(t) - x(t) \) and \( \eta(t) \)
= y_i(t) - y(t), then \((\xi(t), \eta(t)) \neq (0, 0)\). From (3) it follows that \((\xi(t), \eta(t))\) is a solution of

\[
\begin{align*}
\dot{\xi} &= \eta - \phi(x(t), \xi(t)) \\
\dot{\eta} &= h(x(t), \xi(t))
\end{align*}
\]  

(4)

It follows that \((\xi, \eta) = Q(\xi, \eta)\) where \(Q(\xi, \eta) = \eta^2 + h\xi^2 - \phi\xi\eta\), and \(h\) and \(\phi\) are as defined just previous to the statement of the theorem.

Since \(D(x(t), \xi(t)) \leq 0\) for all \(t\), it follows that \((\xi, \eta) \geq 0\); i.e., \(\xi\eta\) is nondecreasing in \(t\). But since \(\xi\) and \(\eta\) are, by hypothesis, bounded, it follows that there exist numbers \(c\) and \(d\) such that \(c \leq d\), that \(\xi\eta \to c\) as \(t \to -\infty\), and that \(\xi\eta \to d\) as \(t \to +\infty\).

Suppose first \(c = d\); then clearly \(\xi\eta = c = d\); hence \((\xi, \eta) = Q(\xi, \eta) = 0\) for all \(t\). But \(Q(\xi, \eta) = 0\) clearly implies \(D(x, \xi) = 0\), and from the hypotheses of the theorem, \(\xi = 0\). But then \(Q(0, \eta) = 0\) implies \(\eta = 0\). Hence \(c = d\) implies \((\xi, \eta) = (0, 0)\), a contradiction.

Suppose \(c < d\); \(d \neq 0\). Since \(\xi\eta \to d\) as \(t \to +\infty\), there exists a sequence \(\{t_n\}, n = 1, 2, \ldots, t_n \to +\infty\), such that for \(t = t_n \to +\infty\), \((\xi, \eta) \to 0\); i.e., \(Q(\xi(t_n), \eta(t_n)) \to 0\) as \(n \to \infty\). Since \(\eta(t)\) is bounded for all \(t\), \(\xi\eta \to d\neq 0\) implies that \(\xi(t_n)\) cannot approach zero as \(n \to \infty\). Hence, there exists a subsequence \(\{t_{n_i}\}\) of \(\{t_n\}\) such that \(t_{n_i} \to +\infty\) as \(i \to \infty\), and that \(|\xi(t_{n_i})| \geq \rho > 0\) for some constant \(\rho\). But \(Q(\xi, \eta) \to 0\) implies \(D(x, \xi) \to 0\) is \(i \to \infty\), here \(x_i = x(t_{n_i})\), and \(\xi_i\) and \(\eta_i\) are similarly defined. Hence by Remark 2, \(\xi_i \to 0\) as \(i \to \infty\), a contradiction.

Finally suppose \(d = 0, c < 0\). Since \(\xi\eta \to d\) as \(t \to +\infty\), there exists a sequence \(\{t_n\}, n = 1, 2, \ldots, t_n \to -\infty\), such that \(Q(\xi(t_n), \eta(t_n)) \to 0\) and \(n \to \infty\). Again the fact that \(\xi(t)\) cannot approach zero on any subsequence of \(\{t_n\}\) leads to a contradiction as before; we omit the details.

Hence, the solution \((x(t), y(t)) \in R\) for all \(t\) is unique, and the lemma is proved.

**Lemma 5.** If \(\Gamma_+\) and \(\Gamma_1\) are as defined in Lemmas 1 and 2, and if \((x(t), y(t))\) is the unique solution of (3) such that \((x(t), y(t)) \in R\) for all \(t\), then there exist nonempty closed sets \(\Gamma_+ \subset \Gamma_1\) and \(\Gamma_- \subset \Gamma_+\) such that if \((x_1(t), y_1(t))\) is a solution of (3) and \((\xi(t), \eta(t))\) are as defined in Lemma 4, then

(a) \((x_1(0), y_1(0)) \in \Gamma_+\) implies \((\xi(t), \eta(t)) \to (0, 0)\) as \(t \to +\infty\);

(b) \((x_1(0), y_1(0)) \in \Gamma_-\) implies \((\xi(t), \eta(t)) \to (0, 0)\) as \(t \to -\infty\).

**Proof.** From the proof of Lemma 2 it is clear that a closed nonempty subset \(\Gamma_+\) of \(\Gamma_1\) exists such that if \((x_1(0), y_1(0)) \in \Gamma_+\), then \((x_1(t), y_1(t)) \in R\) for \(t > 0\). Similarly, a subset \(\Gamma_-\) of \(\Gamma_+\) exists such that
if \((x_1(0), y_1(0)) \in \Gamma\), then \((x_1(t), y_1(t)) \in \mathbb{R}\) for \(t < 0\); we omit the details.

We first prove (a). As in the proof of Lemma 4, it follows that \(\xi \eta\) is a nondecreasing function of \(t\) for \(t > 0\), and, in fact, \(\xi \eta \to 0\) as \(t \to +\infty\). (In this paragraph all limits are as \(t \to +\infty\).) Hence \(\xi \eta \leq 0\) for \(t > 0\). But since \((\xi, \eta)\) satisfies (4), we have, since \(\phi \geq 0, \xi \xi = \xi \eta - \phi \xi^2 \leq 0\). Hence \(\xi^2\) is a nonincreasing function of \(t\), and thus \(\xi^2 \to \lambda_1^2\) where \(\lambda_1 \geq 0\). Suppose \(\lambda_1 > 0\); then \(|\xi| \to \lambda_1\), and in fact \(|\xi| \geq \lambda_1\) for \(t > 0\). From the properties of \(h(x, \xi)\), it follows that there exists a constant \(\epsilon > 0\) such that \(h(x(t), \xi(t)) \geq \epsilon\) for \(t > 0\). Hence from (4), \(|\eta| \geq \epsilon \lambda_1\), and we conclude that \(|\eta|\) cannot be bounded for \(t > 0\) as it must be by hypothesis. Hence \(\lambda_1 = 0\); i.e., \(\xi \to 0\). From (4) we also have \(\eta \eta = h \xi \xi \leq 0\); hence \(\eta^2\) is nonincreasing and therefore \(|\eta| \to \lambda_2 \geq 0\).

If \(\lambda_2 > 0\), then from (4) and the fact that \(\xi \to 0\), we have \(|\xi| \to \lambda_2\). This again leads to a contradiction since \(|\xi|\) is bounded for \(t > 0\). Hence \(\lambda_2 = 0\), and finally \((\xi, \eta) \to (0, 0)\).

To prove (b) we first make a change of variable \(r = -t\) in (4); hence (4) becomes

\[
\begin{align*}
\dot{\xi} &= -\eta + \phi \xi, \\
\dot{\eta} &= -h \xi
\end{align*}
\]

where \(\xi = \xi(-r), \eta = \eta(-r)\), and the dot now indicates differentiation with respect to \(r\). Then in what follows in this paragraph, all limits are as \(r \to +\infty\). Since \((\xi', \eta) = -Q(\xi, \eta)\) where \(Q\) is as in Lemma 4, it follows that \(\xi \eta\) is nonincreasing in \(r\), and that, as in the previous case, \(\xi \eta \to 0\). Hence \(\xi \eta \geq 0\) for \(r > 0\). From (4'), \(\eta \eta = -h \xi \xi \leq 0\); hence \(\eta^2\) is nonincreasing, and \(|\eta| \to \mu \geq 0\). If \(\mu > 0\), then \(\xi \to 0\), and from (4'), \(|\xi| \to \mu\) which again contradicts the boundedness of \(|\xi|\) for \(r > 0\). Hence \(\mu = 0\); i.e., \(\eta \to 0\). Suppose now \(\xi \to 0\). Then there exist \(\delta > 0\) and \(\tau_0 > 0\) such that \(|\xi_0| = \delta\), and if \(\delta_1\) is such that \(|\phi \xi| \geq 2\delta_1 > 0\) for \(|\xi| \geq \delta_1\), then \(|\eta| < \delta_1\); here \(\xi_0 = \xi(-\delta_0)\). Suppose \(\xi_0 = \delta\); then from (4'), \(\xi = -\eta + \phi \xi > \delta_1\) when \(r = \tau_0\). We now show that \(\xi > \xi_0\) for \(\tau > \tau_0\). If not, then \(\xi = \xi_0\) for \(\tau = \tau_2 > \tau_0\) while \(\xi > \xi_0\) for \(\tau_0 < \tau < \tau_2\). Hence, there exists a \(\tau_1, \tau_0 < \tau_1 < \tau_2\), such that for \(\tau = \tau_1, \xi = 0\). Thus by (4'), \(-\eta + \phi \xi = 0\) for \(\tau = \tau_1\). But \(-\eta + \phi \xi > \delta_1\) for \(\tau_0 < \tau < \tau_2\) and we have a contradiction. This proves that \(\xi > \delta\) for \(\tau > \tau_0\). Hence, using (4'), we have \(\xi > \delta_1\) for \(\tau > \tau_0\) and thus again a contradiction. Hence if \(\xi_0 = \delta\), then \(\xi \to 0\). The argument for the case \(\xi_0 = -\delta\) proceeds similarly; we omit the details. Hence in any case \((\xi, \eta) \to (0, 0)\) as \(r \to \infty\) and the proof of the lemma is complete.

Proof of theorem. From Lemmas 3 and 4 and a general result due to Amerio [3] which states that the unique solution \((x(t), y(t))\)
of (3) such that \((x(t), y(t)) \in R\) for all \(t\) is, in fact, a.p., we conclude the existence of a unique a.p. solution of (3). From Lemma 5, it follows that we may take as \(\Sigma_1\) the set of solutions \((x(t), y(t))\) of (3) such that \((x(0), y(0)) \in \Gamma_+\) and as \(\Sigma_2\) the set of \((x(t), y(t))\) such that \((x(0), y(0)) \in \Gamma_-\). Clearly (2) is a system of the form of (3). Hence the theorem follows.

We observe that there exist closed subsets of \(\Gamma_b\) and \(\Gamma_2 = \Gamma - (\Gamma_1 \cup \Gamma_a \cup \Gamma_b)\) having the same properties as the sets \(\Gamma_-\) and \(\Gamma_+\) respectively of Lemma 5.

3. **Extension of the theorem.** The part of condition (iii) requiring \(g(x) > k\) for \(x \geq b\) and \(g(x) < -k\) for \(x \leq a\) can be weakened. For example, suppose there exists a region \(S\) of the \((x, y)\) plane such that \(R \cap S\) is empty, and such that the graph of \(y = F(x)\) for \(x_0 < x < x_1\) is contained in \(S\); here \(x_0 = -\infty\) or \(x_1 = +\infty\) is allowed. Suppose further that for any solution \((x(t), y(t))\) of (2) if \((x(t_0), y(t_0)) \in S\), then \((x(t), y(t)) \in S\) for \(t > t_0\). Then the theorem will follow even if possibly \(|g(x)| \leq k\) for \(x_0 < x < x_1\). For again, any trajectory of (3) leaving \(R\) at some \(t = t_0\) will not return to \(R\) for any \(t > t_0\), and Lemma 1 and all subsequent lemmas would again follow.

**References**


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