ALMOST PERIODIC SOLUTIONS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS NEAR POINTS OF NONLINEAR FIRST APPROXIMATION

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1. Introduction. Sufficient conditions for the existence of almost periodic (a.p. for short) solutions of systems of the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + r(x_1, \cdots, x_n, t), \quad i = 1, 2, \cdots, n, \tag{1}$$

are known provided the characteristic values of the matrix $a=(a_{ij})$ have nonzero real parts; cf., for example, [1]. Here the r_i are uniformly a.p. in t for $(x_i, \cdots, x_n)$ in some region of Euclidean n-space. Amerio [3] has obtained such conditions for somewhat more specialized systems where some characteristic values of A may be pure imaginary; however, all must be distinct from zero.

In case $n=2$, sufficient conditions for the existence, uniqueness, and asymptotic stability of a.p. solutions are known even if one of the characteristic values of A is zero. In fact, the system considered in [2] is

$$\dot{x} = y - F(x),$$
$$\dot{y} = -g(x) + p(t),$$

where $F(0)=g(0)=0$, and the derivatives $F'(x)$ and $g'(x)$ are such that $F'(x)>0$, and $g'(x)\geqq 0$ where $g'(x)=0$ only when $x=0$. In this paper, a system of essentially this type is considered; we require now that $g'(x)\leqq 0$ where $g'(x)=0$ only when $x=0$, and if $g'(0)=0$, then $F'(0)=0$. We prove an existence theorem for a unique a.p. solution which in case $F'(x)>0$ for $x\neq 0$, has certain stability properties. We observe that this system is equivalent to the single second order equation:

$$\ddot{x} + F'(x)\dot{x} + g(x) = p(t).$$

Throughout this paper, conventional topological definitions and notation is used; i.e., a region is an open connected plane set; the closure, union, intersection of sets are denoted respectively by $\overline{R}$, $R\cup S$, $R\cap S$ where R and S are sets; $p\in S$ means p is a member of S, etc.

Received by the editors July 15, 1959.

[1] Sponsored by the Office of Ordnance Research, U. S. Army.

2. **The existence theorem.** We consider the system

$$\dot{x} = y - F(x),$$
$$(2) \qquad \dot{y} = g(x) + p(t);$$

here (i) $p(t)$ is a.p. and $k = \max |p(t)|$;

(ii) $g(0) = F(0) = 0$;

(iii) there exist numbers a, b, $a < 0 < b$, such that $g(x) > k$ for $x \geqq b$, $g(x) < -k$ for $x \leqq a$;

(iv) $g'(x)$ and $F'(x)$ are continuous, and $g'(x) \geqq 0$ for $a \leqq x \leqq b$.

We define

$$\phi(x, \xi) = \begin{cases} (F(x + \xi) - F(x))/\xi, & \xi \neq 0, \\ F'(x), & \xi = 0; \end{cases}$$

$$h(x, \xi) = \begin{cases} (g(x + \xi) - g(x))/\xi, & \xi \neq 0, \\ g'(x), & \xi = 0; \end{cases}$$

and

$$D(x, \xi) = \phi^2(x, \xi) - 4h(x, \xi).$$

In terms of these definitions we state and prove the following

THEOREM. *If $D(x, \xi) \leqq 0$ for $a \leqq x \leqq b$, $a \leqq x + \xi \leqq b$, and $D(x, \xi) = 0$ only if $\xi = 0$, then system* (2) *has a unique a.p. solution. If also $\phi(x, \xi) > 0$ for $\xi \neq 0$, then this a.p. solution is asymptotically stable as $t \to +\infty$ with respect to a class Σ_1 of solutions of* (2), *and asymptotically stable as $t \to -\infty$ with respect to a class Σ_2 of solutions of* (2).

The proof of this theorem is based on a series of lemmas which are stated and proved after the following remarks.

REMARK 1. $g'(0) = F'(0) = 0$ is possible; hence, if (2) is considered as a special case of (1), the matrix A is in that case given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the characteristic values of which are both zero. Hence, the system is markedly nonlinear at the origin.

REMARK 2. If $a \leqq x_n \leqq b$, $a \leqq x_n + \xi_n \leqq b$ for $n = 1, 2, \cdots$, then $D(x_n, \xi_n) \to 0$ as $n \to \infty$ implies $\xi_n \to 0$ as $n \to \infty$; this clearly follows from the hypotheses of the theorem, and continuity.

The following lemmas concern themselves with the system

$$\dot{x} = y - F(x),$$
$$(3) \qquad \dot{y} = g(x) + e(t),$$

where $e(t)$ is in the closure of the set of functions $\{p(t+h)\}$, $|h| < \infty$, the closure being with respect to the uniform norm on $|t| < \infty$. Clearly $|e(t)| \leqq k$ for all t. Conditions on $g(x)$ and $F(x)$ as before (involving k) are presumed.

We consider the region R of the (x, y) plane bounded by the lines $y = mx \pm c$, $x = a$, $x = b$; here $m > 0$ is fixed but arbitrary, and $c > 0$ is such that for $a \leqq x \leqq b$ and $n = 1, 2$:

$$\text{(v)} \qquad m > \frac{g(x) + (-1)^n k}{(-1)^n c + mx - F(x)},$$

and

$$\text{(vi)} \qquad | mx + (-1)^n c | > F(x).$$

We denote by Γ the boundary of R, and by Γ_a and Γ_b the subsets of Γ respectively defined by $x = a$, $ma - c < y < F(a)$ and $x = b$, $F(b) < y < mb + c$.

LEMMA 1. *Let $(x(t), y(t))$ be a solution of* (3) *such that $(x(t_0), y(t_0)) \in R$ and $(x(t_2), y(t_2)) \notin \overline{R}$ for some $t_2 > t_0$. Then there exists a t_1, $t_0 < t_1 < t_2$, such that $(x(t_1), y(t_1)) \in \Gamma_a \cup \Gamma_b$, and such that $(x(t), y(t)) \notin R$ for $t > t_1$. Also, either $x(t_2) < a$ or $x(t_2) > b$, in the former case, $(x(t_1), y(t_1)) \in \Gamma_a$, in the latter case $(x(t_1), y(t_1)) \in \Gamma_b$.*

PROOF. From (vi) above, the graph of $y = F(x)$ for $a < x < b$ is in R. It follows from (v) that the slope of Γ along the lines $y = mx \pm c$ is greater than the slope of the trajectory of any solution of (3) at points on these parts of Γ, i.e., trajectories of (3) cannot leave R there as t increases.

Since $y > F(x)$ implies $\dot{x} > 0$ and $y < F(x)$ implies $\dot{x} < 0$, it follows that trajectories of (3) above $y = F(x)$ move to the right as t increases, while those below $y = F(x)$ move to the left. Hence, trajectories cannot leave R on the part of Γ on $x = a$ above $y = F(x)$ and on $x = b$ below $y = F(x)$.

It remains only to consider the trajectories at points $(a, F(a))$ and $(b, F(b))$ of Γ. In fact, if the trajectory of a solution $(x(t), y(t))$ of (3) is such that $x(t_0) = b_1 \geqq b$, $y(t_0) = F(b_1)$, then $\dot{x}(t_0) = 0$ and $\dot{y}(t_0) > 0$. Hence this trajectory has a vertical tangent at $(b_1, F(b_1))$, is on the right side of this tangent for $t \neq t_0$, and moves up across the graph of $y = F(x)$ there as t increases. It follows that any trajectory leaving R on Γ_b cannot return to R for t sufficiently large.

A similar discussion applies to the situation at points of the form $(a_1 F(a_1))$ where $a_1 \leqq a$, except that at such points, trajectories of (3) cross down over $y=F(x)$ as t increases; we omit the details, and the proof of the lemma is complete.

LEMMA 2. *There exists a solution* $(x(t), y(t))$ *of* (3) *such that* $(x(t), y(t))\in\overline{R}$ *for all* $t\geqq 0$.

PROOF. Let p be a point of (x, y) plane; then we will denote by $\mu(p, t)$ the point $(x(t), y(t))$ where $(x(t), y(t))$ is the solution of (3) such that $(x(0), y(0))=p$. We also denote by p_1 and p_2 the points of Γ given by $(a, F(a))$ and $(b, mb+c)$ respectively and by Γ_1 the part of Γ between these points but above the curve $y=F(x)$. Let the set of points on Γ_1 be ordered so that if p and q are distinct points on Γ_1, then $p>q$ means that the distance along Γ_1 from p to p_2 is greater than the distance along Γ_1 from q to p_2.

Consider the set S_b of points on Γ_1 such that if $p\in S_b$, then for each $q\in\Gamma_1$, $q<p$, there exists a $T=T(q)\geqq 0$ such that $\mu(q, T)\in\Gamma_b$. This set is clearly nonempty and in fact contains an open interval of Γ_1 which has p_2 as its end point. Similarly, Γ_1-S_b contains an interval having p_1 as end point. Since S_b is bounded above, it has a least upper bound (l.u.b. for short), say $q_1\neq p_1$. Clearly $q_1\neq p_2$. We shall show that $\mu(q_1, t)\in R$ for all $t>0$. For if not, then by Lemma 1 there exists a $T_1\geqq 0$ such that $\mu(q_1, T_1)\in\Gamma_a\cup\Gamma_b$. Suppose $\mu(q_1, T_1)\in\Gamma_a$, and let $t_1>0$ be fixed but arbitrary. Then by Lemma 1 there exists a region T_a in the half plane $x<a$ such that $\mu(q_1, T_1+t_1)\in R_a$. By continuity of solutions of (3) as functions of their initial conditions, there exists a region S_1 such that $q_1\in S_1$, and if $p\in S_1\cap\Gamma_1$, there $\mu(p, T_1+t_1)\in R_a$, and hence, by Lemma 1 there exists a θ, $0<\theta<1$, such that $\mu(p, \theta(T_1+t_1))\in\Gamma_a$. Since $S_1\cap\Gamma_1$ is an open interval of Γ_1 containing q_1, this contradicts the assumption that q_1 is a l.u.b. for S_b.

The case $\mu(q_1, T_2)\in\Gamma_b$ for some $T_2\geqq 0$ is ruled out in the same way; we omit the details.

LEMMA 3. *There exists a solution* $(x(t), y(t))$ *of* (3) *such that* $(x(t), y(t))\in R$ *for all* t.

PROOF. From Lemma 2, there exists a solution $(x_0(t), y_0(t))$ of (3) such that $(x(t), y(t))\in\overline{R}$ for $t\geqq 0$. From a result of Amerio [**3**], there exists a solution $(x(t), y(t))$ of (3) such that $(x(t), y(t))\in R$ for all t. This proves the lemma.

LEMMA 4. *If* $(x(t), y(t))\in R$ *for all* t, *then this solution is unique.*

PROOF. Suppose there exists a solution $(x_1(t), y_1(t))$ of (3) such that $(x_1(t), y_1(t))\in R$ for all t, and such that if $\xi(t)=x_1(t)-x(t)$ and $\eta(t)$

$=y_1(t)-y(t)$, then $(\xi(t),\ \eta(t))\neq(0,\ 0)$. From (3) it follows that $(\xi(t),\ \eta(t))$ is a solution of

$$\begin{aligned} \dot{\xi} &= \eta - \phi(x(t),\ \xi)\xi, \\ \dot{\eta} &= h(x(t),\ \xi)\xi. \end{aligned} \tag{4}$$

It follows that $(\xi\dot{}\eta)=Q(\xi,\ \eta)$ where $Q(\xi,\ \eta)=\eta^2+h\xi^2-\phi\xi\eta$, and h and ϕ are as defined just previous to the statement of the theorem.

Since $D(x(t),\ \xi(t))\leqq 0$ for all t, it follows that $(\xi\dot{}\eta)\geqq 0$; i.e., $\xi\eta$ is nondecreasing in t. But since ξ and η are, by hypothesis, bounded, it follows that there exist numbers c and d such that $c\leqq d$, that $\xi\eta\rightarrow c$ as $t\rightarrow-\infty$, and that $\xi\eta\rightarrow d$ as $t\rightarrow+\infty$.

Suppose first $c=d$; then clearly $\xi\eta=c=d$; hence $(\xi\dot{}\eta)=Q(\xi,\ \eta)=0$ for all t. But $Q(\xi,\ \eta)=0$ clearly implies $D(x,\ \xi)=0$, and from the hypotheses of the theorem, $\xi=0$. But then $Q(0,\ \eta)=0$ implies $\eta=0$. Hence $c=d$ implies $(\xi,\ \eta)=(0,\ 0)$, a contradiction.

Suppose $c<d$, $d\neq 0$. Since $\xi\eta\rightarrow d$ as $t\rightarrow+\infty$, there exists a sequence $\{t_n\}$, $n=1,\ 2,\ \cdots,\ t_n\rightarrow+\infty$, such that for $t=t_n\rightarrow+\infty$, $(\xi\dot{}\eta)\rightarrow 0$; i.e., $Q(\xi(t_n),\ \eta(t_n))\rightarrow 0$ as $n\rightarrow\infty$. Since $\eta(t)$ is bounded for all t, $\xi\eta\rightarrow d\neq 0$ implies that $\xi(t_n)$ cannot approach zero as $n\rightarrow\infty$. Hence, there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i}\rightarrow+\infty$ as $i\rightarrow\infty$, and that $|\xi(t_{n_i})|\geqq\rho>0$ for some constant ρ. But $Q(\xi_i,\ \eta_i)\rightarrow 0$ implies $D(x_i,\ \xi_i)\rightarrow 0$ is $i\rightarrow\infty$, here $x_i=x(t_{n_i})$, and ξ_i and η_i are similarly defined. Hence by Remark 2, $\xi_i\rightarrow 0$ as $i\rightarrow\infty$, a contradiction.

Finally suppose $d=0$ and $c<0$. Then there exists a sequence $\{t_n\}$, $n=1,\ 2,\ \cdots,\ t_n\rightarrow-\infty$, such that $Q(\xi(t_n),\ \eta(t_n))\rightarrow 0$ and $n\rightarrow\infty$. Again the fact that $\xi(t)$ cannot approach zero on any subsequence of $\{t_n\}$ leads to a contradiction as before; we omit the details.

Hence, the solution $(x(t),\ y(t))\in R$ for all t is unique, and the lemma is proved.

Lemma 5. *If* Γ_a *and* Γ_1 *are as defined in Lemmas* 1 *and* 2, *and if* $(x(t),\ y(t))$ *is the unique solution of* (3) *such that* $(x(t),\ y(t))\in R$ *for all* t, *then there exist nonempty closed sets* $\Gamma_+\subset\Gamma_1$ *and* $\Gamma_-\subset\Gamma_a$ *such that if* $(x_1(t),\ y_1(t))$ *is a solution of* (3) *and* $(\xi(t),\ \eta(t))$ *are as defined in Lemma* 4, *then*

(a) $(x_1(0),\ y_1(0))\in\Gamma_+$ *implies* $(\xi(t),\ \eta(t))\rightarrow(0,\ 0)$ as $t\rightarrow+\infty$;

(b) $(x_1(0),\ y_1(0))\in\Gamma_-$ *implies* $(\xi(t),\ \eta(t))\rightarrow(0,\ 0)$ *as* $t\rightarrow-\infty$.

Proof. From the proof of Lemma 2 it is clear that a closed nonempty subset Γ_+ of Γ_1 exists such that if $(x_1(0),\ y_1(0))\in\Gamma_+$, then $(x_1(t),\ y_1(t))\in R$ for $t>0$. Similarly, a subset Γ_- of Γ_a exists such that if $(x_1(0), y_1(0)) \in \Gamma$, then $(x_1(t), y_1(t)) \in R$ for $t<0$; we omit the details.

We first prove (a). As in the proof of Lemma 4, it follows that $\xi\eta$ is a nondecreasing function of t for $t>0$, and, in fact, $\xi\eta \to 0$ as $t \to +\infty$. (In this paragraph all limits are as $t \to +\infty$.) Hence $\xi\eta \leqq 0$ for $t>0$. But since (ξ, η) satisfies (4), we have, since $\phi \geqq 0$, $\xi\dot{\xi} = \xi\eta - \phi\xi^2 \leqq 0$. Hence ξ^2 is a nonincreasing function of t, and thus $\xi^2 \to \lambda_1^2$ where $\lambda_1 \geqq 0$. Suppose $\lambda_1 > 0$; then $|\xi| \to \lambda_1$, and in fact $|\xi| \geqq \lambda_1$ for $t>0$. From the properties of $h(x, \xi)$, it follows that there exists a constant $\epsilon > 0$ such that $h(x(t), \xi(t)) \geqq \epsilon$ for $t>0$. Hence from (4), $|\dot{\eta}| \geqq \epsilon\lambda_1$, and we conclude that $|\eta|$ cannot be bounded for $t>0$ as it must be by hypothesis. Hence $\lambda_1 = 0$; i.e., $\xi \to 0$. From (4) we also have $\eta\dot{\eta} = h\xi\eta \leqq 0$; hence η^2 is nonincreasing and therefore $|\eta| \to \lambda_2 \geqq 0$. If $\lambda_2 > 0$, then from (4) and the fact that $\xi \to 0$, we have $|\dot{\xi}| \to \lambda_2$. This again leads to a contradiction since $|\xi|$ is bounded for $t>0$. Hence $\lambda_2 = 0$, and finally $(\xi, \eta) \to (0, 0)$.

To prove (b) we first make a change of variable $\tau = -t$ in (4); hence (4) becomes

$$\begin{aligned} \dot{\xi} &= -\eta + \phi\xi, \\ \dot{\eta} &= -h\xi \end{aligned} \tag{4'}$$

where $\xi = \xi(-\tau)$, $\eta = \eta(-\tau)$, and the dot now indicates differentiation with respect to τ. Then in what follows in this paragraph, all limits are as $\tau \to +\infty$. Since $(\xi\dot{}\eta) = -Q(\xi, \eta)$ where Q is as in Lemma 4, it follows that $\xi\eta$ is nonincreasing in τ, and that, as in the previous case, $\xi\eta \to 0$. Hence $\xi\eta \geqq 0$ for $\tau > 0$. From (4′), $\eta\dot{\eta} = -h\xi\eta \leqq 0$; hence η^2 is nonincreasing, and $|\eta| \to \mu \geqq 0$. If $\mu > 0$, then $\xi \to 0$, and from (4′), $|\dot{\xi}| \to \mu$ which again contradicts the boundedness of $|\xi|$ for $\tau > 0$. Hence $\mu = 0$; i.e., $\eta \to 0$. Suppose now $\xi \not\to 0$. Then there exist $\delta > 0$ and $\tau_0 > 0$ such that $|\xi_0| = \delta$, and if δ_1 is such that $|\phi\xi| \geqq 2\delta_1 > 0$ for $|\xi| \geqq \delta$, then $|\eta| < \delta_1$; here $\xi_0 = \xi(-\tau_0)$. Suppose $\xi_0 = \delta$; then from (4′), $\dot{\xi} = -\eta + \phi\xi > \delta_1$ when $\tau = \tau_0$. We now show that $\xi > \xi_0$ for $\tau > \tau_0$. If not, then $\xi = \xi_0$ for $\tau = \tau_2 > \tau_0$ while $\xi > \xi_0$ for $\tau_0 < \tau < \tau_2$. Hence, there exists a τ_1, $\tau_0 < \tau_1 < \tau_2$, such that for $\tau = \tau_1$, $\dot{\xi} = 0$. Thus by (4′), $-\eta + \phi\xi = 0$ for $\tau = \tau_1$. But $-\eta + \phi\xi > \delta_1$ for $\tau_0 < \tau < \tau_2$ and we have a contradiction. This proves that $\xi > \delta$ for $\tau > \tau_0$. Hence, using (4′), we have $\dot{\xi} > \delta_1$ for $\tau > \tau_0$ and thus again a contradiction. Hence if $\xi_0 = \delta$, then $\xi \to 0$. The argument for the case $\xi_0 = -\delta$ proceeds similarly; we omit the details. Hence in any case $(\xi, \eta) \to (0, 0)$ as $\tau \to \infty$ and the proof of the lemma is complete.

Proof of theorem. From Lemmas 3 and 4 and a general result due to Amerio [3] which states that the unique solution $(x(t), y(t))$

of (3) such that $(x(t), y(t)) \in R$ for all t is, in fact, a.p., we conclude the existence of a unique a.p. solution of (3). From Lemma 5, it follows that we may take as Σ_1 the set of solutions $(x(t), y(t))$ of (3) such that $(x(0), y(0)) \in \Gamma_+$ and as Σ_2 the set of $(x(t), y(t))$ such that $(x(0), y(0)) \in \Gamma_-$. Clearly (2) is a system of the form of (3). Hence the theorem follows.

We observe that there exist closed subsets of Γ_b and $\Gamma_2 = \Gamma - (\Gamma_1 \cup \Gamma_a \cup \Gamma_b)$ having the same properties as the sets Γ_- and Γ_+ respectively of Lemma 5.

3. **Extension of the theorem.** The part of condition (iii) requiring $g(x) > k$ for $x \geqq b$ and $g(x) < -k$ for $x \leqq a$ can be weakened. For example, suppose there exists a region S of the (x, y) plane such that $R \cap S$ is empty, and such that the graph of $y = F(x)$ for $x_0 < x < x_1$ is contained in S; here $x_0 = -\infty$ or $x_1 = +\infty$ is allowed. Suppose further that for any solution $(x(t), y(t))$ of (2) if $(x(t_0), y(t_0)) \in S$, then $(x(t), y(t)) \in S$ for $t > t_0$. Then the theorem will follow even if possibly $|g(x)| \leqq k$ for $x_0 < x < x_1$. For again, any trajectory of (3) leaving R at some $t = t_0$ will not return to R for any $t > t_0$, and Lemma 1 and all subsequent lemmas would again follow.

References

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