A DISCRETE ANALOGUE OF THE WEIERSTRASS TRANSFORM[1]

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1. Introduction and definitions. In this note we consider a discrete analogue of the transform

$$f(x) = \int_{-\infty}^{\infty} e^{-(x-y)^2/4} g(y)dy.^2$$

This transform may be considered as a convolution; accordingly our analogue will take the form

$$f(n) = \sum_{-\infty}^{\infty} k(n-m)g(m)$$

where the kernel sequence $k(n)$ is to be appropriately chosen. The choice of $k(n)$ is motivated as follows.

One way in which the kernel $e^{-v^2/4}$ arises in the theory of the convolution transform is by consideration of the class of kernels whose bilateral Laplace transform is of the form

$$e^{cs^2}e^{bs}\prod_{1}^{\infty}\left(1-\frac{s}{a_k}\right)^{-1}e^{-s/a_k}. \tag{1.1}$$

When $c=0$ we obtain the class of kernels whose analogue was considered by Pollard and Standish [4] where the kernels formed a subclass of the totally positive sequences. The totally positive sequences can be characterized as sequences having a generating function of the form

$$e^{az+bz^{-1}}z^m\prod_{1}^{\infty}\frac{(1+\alpha_j z)(1+\beta_j z^{-1})}{(1-a_j z)(1-b_j z^{-1})} \tag{1.2}$$

(see [1]). The factor $e^{az+bz^{-1}}$ in (1.2) corresponds to the factor e^{cs^2} in (1.1).

Letting $a=b=v/2$ we have the familiar expansion

$$e^{(v/2)(z+z^{-1})} = \sum_{-\infty}^{\infty} I_m(v)z^m$$

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[1] This work is part of a doctoral thesis done under the direction of Professor Harry Pollard at Cornell University.

[2] See Hirschman and Widder [3] for the theory of the Weierstrass transform.

where $I_m(v)$ is the modified Bessel coefficient of the first kind.

We take the sequence $\{I_n(1)\}_{n=-\infty}^{\infty} = \{I_n\}_{-\infty}^{\infty}$ as the kernel of our transform

$$f(n) = \sum_{-\infty}^{\infty} I_{n-m} g(m). \tag{1.3}$$

Before proceeding further we list some elementary properties of the functions $I_n(v)$.

(i) $$I_n(v) = \sum_{0}^{\infty} \frac{(v/2)^{n+2s}}{s!(s+n)!}, \qquad n = 0, \pm 1, \pm \cdots$$

(ii) $$I_n(v) = I_{-n}(v); \quad I_n(-v) = (-1)^n I_n(v),$$

(iii) $$\frac{2(1/2)^{|n|}}{|n|!} \leqq I_n(1) \leqq \frac{3(1/2)^{|n|}}{|n|!},$$

(iv) $$I_n(v+v') = \sum_{-\infty}^{\infty} I_{n-m}(v) I_m(v').$$

If we take $g(m) = 2^{|m|}|m|!/(1+m^2)$ it is clear, by (iii), that (2.13) converges only for the value $n=0$. We therefore define the transform (1.3) only when it converges for all n.

Define the operators δ and δ^{-1} by the equations

$$\delta f(n) = f(n-1), \qquad \delta^{-1} f(n) = f(n+1).$$

Then by analogy with the theory of the Weierstrass transform we may expect (1.3) to be inverted by the operator $e^{-(1/2)(\delta+\delta^{-1})}$ interpreted as

$$[e^{-(1/2)(\delta+\delta^{-1})}]f(n) = \lim_{t\to 0+} \sum_{-\infty}^{\infty} (-1)^m I_m f(n-m)/\Gamma(1+tm). \tag{1.4}$$

The method of summation adopted in (1.4) is a regular method introduced by Mittag-Leffler (see [2; p. 72]). The motivation for choosing Mittag-Leffler summation as the "natural" method for the inversion operator is indicated in the table below.

	Continuous	Discrete
Kernel	$e^{-y^2/4}$	$I_m\ 1/m!$
Convergence factor	$e^{-(1/t-1)y^2/4}$ $t\to 1-$	$1/\Gamma(1+t\lvert m\rvert)$ $t\to 0+$

Unfortunately the operator (1.4) will not serve to invert all convergent transforms (1.3). For example let

$$g(m) = \begin{cases} \dfrac{2^m m!}{m^{\log m}} & m > 1, \\ 0 & m \leqq 1. \end{cases}$$

Then, if $n>0$ say, we have

$$f(n) = \sum_1^\infty I_{n-m} \frac{2^m m!}{m^{\log m}} > \sum_1^\infty \frac{2^m m!}{2^{|n-m|} \,|\, n-m \,|\, ! m^{\log m}}$$
$$> \sum_n^\infty \frac{2^n m!}{(m-n)! m^{\log m}} > \sum_n^\infty \frac{m^n}{m^{\log m}}.$$

If in the last sum above we consider only the term with $m = [e^{n/2}]$ we obtain

$$f(n) > O(e^{n^2/4}) \qquad n \to +\infty;$$

and (1.4) does not converge for any value of t.

In view of the above we restrict attention to the class of determining sequences $g(m)$ which satisfy the condition

$$g(m) = O(|\, m \,|\, ! x^{|m|})$$

as $|m| \to \infty$, where x is strictly less than 2. We abbreviate this restriction by writing $g(m) \in X$.

2. **The inversion theorem.** We first find the order of the transform.

THEOREM 1. *If $g(m) \in X$ then $f(n) = \sum_{-\infty}^{\infty} I_{n-m} g(m)$ converges absolutely for all n, and*

$$f(n) = O(2^{|n|} \,|\, n \,|\, ! \,(2/x - 1)^{-|n|}) \qquad |\, n \,| \to \infty. \tag{2.1}$$

PROOF. The first statement is obvious. To prove the second, note that since the order conditions on $g(m)$ and I_m are the same for $m \to +\infty$ and $m \to -\infty$ the same is true of $f(n)$. It is therefore sufficient to prove the theorem for $n \to +\infty$.

$$|\, f(n) \,| \leqq \sum_{-\infty}^{-1} I_{n-m} |\, g(m) \,| + \sum_0^\infty I_{n-m} |\, g(m) \,| = S' + S.$$

Since $n>0$ it is clear that $S' \leqq \text{const.} \sum_0^\infty (x/2)^m / m^n \to 0$ as $n \to \infty$.

$$S \leqq \text{const. } 2^n \sum_n^\infty \frac{(x/2)^m m!}{(m-n)!} = \text{const. } x^n \sum_0^\infty (m+1) \cdots (m+n)(x/2)^m.$$

The formula

$$\sum_{0}^{\infty} (m+1)\cdots(m+n)y^m = n!/(1-y)^{n+1}, \qquad y<1$$

may be proved by induction. Hence we have

$$S \leqq \text{const.}\frac{x^n n!}{(1-x/2)^n} = \text{const.}\, 2^n n!(2/x-1)^{-n}.$$

THEOREM 2. *If* $g(n)\in X$ *and* $f(n)=\sum_{-\infty}^{\infty} I_{n-m}g(m)$, *then*

$$\lim_{t\to 0+}\sum_{-\infty}^{\infty}(-1)^m I_m f(n-m)/\Gamma(1+t|m|) = g(n). \tag{2.2}$$

PROOF. There is no loss of generality in taking $n=0$. For any $t>0$

$$\begin{aligned}\sum_{-\infty}^{\infty}&(-1)^m I_m f(-m)/\Gamma(1+t|m|)\\ &= \sum_{m=-\infty}^{\infty}(-1)^m I_m/\Gamma(1+t|m|)\sum_{r=-\infty}^{\infty} I_{-m-r}g(r)\\ &= \sum_{r=-\infty}^{\infty} g(r)\sum_{m=-\infty}^{\infty}(-1)^m I_{-m-r}I_m/\Gamma(1+t|m|) = \sum_{r=-\infty}^{\infty} g(r)\mu(r,t).\end{aligned}$$

The interchange of summations is justified by the estimate of Theorem 1.

Since

$$\mu(0,0) = \sum_{-\infty}^{\infty}(-1)^m I_{-m}(1)I_m(1) = \sum_{-\infty}^{\infty} I_{-m}(1)I_m(-1) = I_0(0) = 1$$

(see iv) the theorem will be established if we show

$$\lim_{t\to 0+}\sum_{1}^{\infty} g(r)\mu(r,t) = 0, \tag{2.3}$$

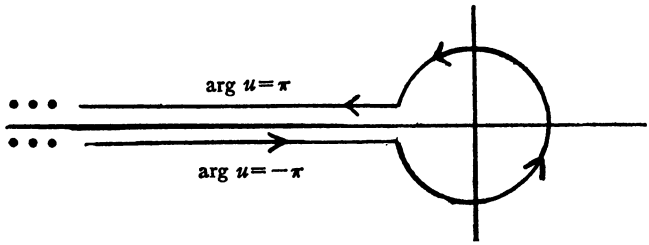
$$\lim_{t\to 0+}\sum_{-\infty}^{-1} g(r)\mu(r,t) = 0. \tag{2.4}$$

We will prove (2.3); the proof of (2.4) is entirely similar.

As a first step let us replace $1/\Gamma(1+t|m|)$ by the contour integral

$$\frac{1}{\Gamma(1+t|m|)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)}\frac{e^u}{u}u_-^{t|m|}du$$

where the path of integration is as shown, the circle being of radius greater than one.



Define the function $K=K(t)=2^{1/t}(2/x-1)^{-1/t}$, so that $K\to+\infty$ as $t\to+$.[3] For any $t>0$ we have

$$\begin{aligned}
2\pi i\sum_{1}^{\infty} g(r)\mu(r,t) &= \sum_{r=1}^{\infty} g(r) \sum_{m=-\infty}^{\infty} (-1)^m I_m I_{m-r} \int_{-\infty}^{(0+)} \frac{e^u}{u} u^{-t|m|} du \\
&= \sum g(r) \sum (-1)^m (\cdots) \int_C (\cdots) du \\
&\quad + \sum g(r) \sum (-1)^m (\cdots) \int_{C_1} (\cdots) du \\
&\quad + \sum g(r) \sum (-1)^m (\cdots) \int_{C_2} (\cdots) du \\
&= S + S_1 + S_2.
\end{aligned}$$

The path C runs from $u=-K(\arg u=-\pi)$ round the circle and back to $u=-K(\arg u=\pi)$, C_1 runs from $-\infty$ to $-K(\arg u=-\pi)$, C_2 runs from $-K$ to $-\infty(\arg u=\pi)$.

We show now that $\lim_{t\to 0+} S_1=0$; the argument showing $\lim_{t\to 0+} S_2=0$ is the same.

$$\begin{aligned}
|S_1| &\leqq \sum_{r=1}^{\infty} |g(r)| \sum_{m=-\infty}^{\infty} I_m I_{m-r} \int_{C_1} \frac{e^u}{u} u^{-t|m|} du \\
&\leqq e^{-K} \sum_{r=1}^{\infty} |g(r)| \sum_{m=-\infty}^{\infty} I_m I_{m-r} K^{-t|m|} = e^{-K} \sum_{m=-\infty}^{\infty} I_m K^{-t|m|} \sum_{r=1}^{\infty} g(r) I_{m-r} \\
&\leqq e^{-K} \sum_{m=-\infty}^{\infty} I_m K^{-t|m|} m! 2^{|m|} \left(\frac{2}{x}-1\right)^{-|m|} = e^{-K} \sum_{-\infty}^{\infty} I_m |m|! = e^{-K} B
\end{aligned}$$

where $B=\sum_{-\infty}^{\infty} I_m |m|!<9$. Therefore $S_1 \leqq e^{-K}B \to 0$ as $t\to 0+$.

[3] It is essential to the proof that $K\to\infty$. This is not the case when $x\leqq 2/3$. But in that event the sum (1.4), *with* $t=0$, converges, and the proof that (1.4) inverts (1.3) is trivial. We therefore assume from now on that $x>2/3$.

The proof has now been reduced to showing that $S \to 0$ as $t \to 0+$. To this end we make use of the formula [5, p. 441]

$$I_m(1)I_{r-m}(1) = \frac{2}{\pi}\int_0^{\pi/2} I_r(2\cos\theta)\cos(2m-r)\theta d\theta.$$

We obtain, setting $u^{-t} = w$,

$$\frac{\pi}{2}S = \sum_{r=1}^{\infty} g(r) \sum_{m=-\infty}^{\infty} (-1)^m \int_0^{\pi/2} I_r(2\cos\theta)\cos(2m-r)\theta d\theta \int_C \frac{e^u}{u} w^{|m|} du$$

$$= \sum_{r=1}^{\infty} g(r)\nu(r, t).$$

Estimate of $\nu(r, t)$. Since $|w| < 1$ along the entire path C, we may write

$$\nu(r, t) = \int_C \frac{e^u}{u}\int_0^{\pi/2} I_r(2\cos\theta)\left\{\sum_{m=-\infty}^{\infty}(-1)^m w^m \cos(2m-r)\theta\right\} d\theta du.$$

$$\sum_{-\infty}^{\infty}(-1)^m w^{|m|}\cos(2m-r)\theta = \cos(r\theta)\frac{1-w^2}{1+2w\cos 2\theta + w^2}.$$

We now have

$$\nu(r, t) = \int_C \frac{e^u}{u}(1-w^2)\int_0^{\pi/2} \frac{I_r(2\cos\theta)\cos r\theta}{1+2w\cos 2\theta + w^2} d\theta du.$$

The integral

$$Q_r(w) = \int_0^{\pi/2} \frac{I_r(2\cos\theta)\cos r\theta}{1+2w\cos 2\theta + w^2} d\theta$$

may be estimated as follows.

Replace $I_r(2\cos\theta)$ by the sum (i) and interchange the order of summation and integration to obtain

$$Q_r(w) = \sum_{s=0}^{\infty} \frac{1}{s!(r+s)!}\int_0^{\pi/2} \frac{(\cos)\,\theta^{r+2s}\cos r\theta}{1+2w\cos 2\theta + w^2} d\theta. \tag{2.5}$$

The integral appearing in (2.5) is now estimated by contour integration. It is easily verified that

$$\int_0^{\pi/2}(\cdots)d\theta = \frac{1}{4}\int_0^{2\pi}(\cdots)d\theta,$$

and so the standard substitution $\cos\theta = 1/2(z+1/z)$, $z = e^{i\theta}$ yields

$$i2^{r+2s+3}\int_0^{\pi/2}(\cdots)d\theta = \int_{|z|=1}\frac{(z^2+1)^{r+2s}(z^{2r}+1)}{z^{2r+2s-1}(wz^4+(1+w^2)z^2+w)}dz$$
$$= \int_{|z|=1}\rho_1(z)dz + \int_{|z|=1}\rho_2(z)dz$$

where

$$\rho_1(z) = \frac{(z^2+1)^{r+2s}}{z^{2s-1}(wz^4+(1+w^2)z^2+w)};$$
$$\rho_2(z) = \frac{(z^2+1)^{r+2s}}{z^{2r+2s-1}(wz^4+(1+w^2)z^2+w)}.$$

Both $\rho_1(z)$ and $\rho_2(z)$ have poles at $z=0$ and at the zeros of $wz^4+(1+w^2)z^2+w$, that is, at $z=\pm iw^{1/2}$, $\pm i/w^{1/2}$. The points $\pm iw^{1/2}$ lie inside the unit circle, while $\pm i/w^{1/2}$ lie outside.

Estimate of $\int_{|z|=1}\rho_1(z)dz$. Let α be a real positive number such that

$$\alpha < \inf_{u\in C}\frac{1}{2}\left|w^{1/2}\right| = \frac{1}{2}\left(\frac{1}{x}-\frac{1}{2}\right)^{1/2}.$$

$$\left|\int_{|z|=1}\rho_1(z)dz\right| \leqq \left|\int_{|z|=\alpha}\rho_1(z)dz\right| + \left|\operatorname{Res}\rho_1(z)\right|_{z=\pm iw^{1/2}}$$

$$\left|\operatorname{Res}\rho_1(z)\right|_{z=+iw^{1/2}} = \frac{\left|1-w\right|^{r+2s}}{2\left|w^s\right|\left|1-w^2\right|},$$

$$\left|\int_{|z|=\alpha}\rho_1(z)dz\right| \leqq 2\pi\alpha^2\frac{(\alpha^2+1)^{r+2s}}{\alpha^{2s}}M_1(w)$$

where $M_1(w)=\sup_{|z|=\alpha}\left|wz^4+(1+w^2)z^2+w^2\right|^{-1}$. $M_1(w)$ certainly exists since the choice of α assures that $wz^4+(1+w^2)z^2+w^2$ is bounded away from zero for $|z|=\alpha$. Now, fix α so small that $x(\alpha^2+1)<2$.

Estimate of $\int_{|z|=1}\rho_2(z)dz$. Let β be a real positive number such that

$$\beta > \sup_{u\in C}2/\left|w^{1/2}\right| = \frac{2}{\left(\frac{1}{x}-\frac{1}{2}\right)^{1/2}},$$

$$\left|\int_{|z|=1}\rho_2(z)dz\right| \leqq \left|\int_{|z|=\beta}\rho_2(z)dz\right| + \left|\operatorname{Res}\rho_2(z)\right|_{z=\pm i/w^{1/2}},$$

$$\left|\operatorname{Res}\rho_2(z)\right|_{z=\pm i/w^{1/2}} = \frac{\left|1-w\right|^{r+2s}}{2\left|w\right|^s\left|1-w^2\right|},$$

$$\left|\int_{z=\beta} \rho_2(z)dz\right| \leqq 2\pi\beta^2 \frac{(\beta^2+1)^{r+2s}}{\beta^{2r+2s}} M_2(w)$$

where $M_2(w)=\sup_{|z|=\beta}|wz^4+(1+w^2)z^2+w|^{-1}$. The choice of β assures that $M_2(w)$ exists. Now fix β so large that $x(\beta^2+1)/\beta^2<2$.

We are now ready to give the estimate of $\nu(r, t)$.

$$|\nu(r,t)| \leqq \int_C \left|\frac{e^u}{u}\right| |1-w^2| \, |Q_r(w)| \, |du|$$

where

$$Q(w) \leqq \sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} \left\{ \frac{|1-w|^{r+2s}}{|w|^s|1-w^2|} + \pi \frac{(\alpha^2+1)^{r+2s}M_1}{\alpha^{2s-2}} + 2\pi \frac{(\beta^2+1)^{r+2s}M_2}{\beta^{2r+2s-2}} \right\}. \tag{2.6}$$

The estimate (2.6) enables us to write

$$\sum_1^{\infty} |g(r)| \, |\nu(r,t)| = \int_C \left|\frac{e^u}{u}\right| |1-w^2| \left\{ \sum_1^{\infty} |g(r)| \, |Q_r(w)| \right\} du. \tag{2.7}$$

The proof is now completed by breaking (2.7) into three parts corresponding to the three sums in (2.6). Call these parts T_1, T_2, and T_3. We give the details showing $T_1 \to 0$ as $t \to 0+$. The details for T_2 and T_3 are similar, and somewhat simpler after observing that $M_1(w)$ and $M_2(w)$ are bounded for $u \in C$ (again by the choice of α and β).

$$T_1 = \int_C \left|\frac{e^u}{u}\right| |1-w^2| \cdot \left\{ \sum_1^{\infty} |g(r)| \sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} \frac{|1-w|^{r+2s}}{|w|^s|1-w^2|} \right\} du.$$

Since $(r+s)! \geqq r!s!$ and since $2^{2s}(s!)^2 \geqq (2s)!$ we have

$$\sum_{s=0}^{\infty} \frac{1}{s!(r+s)!2^{r+2s}} \frac{|1-w|^{r+2s}}{|w|^s|1-w^2|} \leqq \frac{|1-w|^r}{|1-w^2|r!2^r} \sum_{s=0}^{\infty} \frac{|1-w|^{2s}}{(|w|^{1/2})^{2s}} \frac{1}{(2s)!} \leqq \frac{|1-w|^r}{|1-w^2|r!2^r} e^{|1-w|/|w|^{1/2}},$$

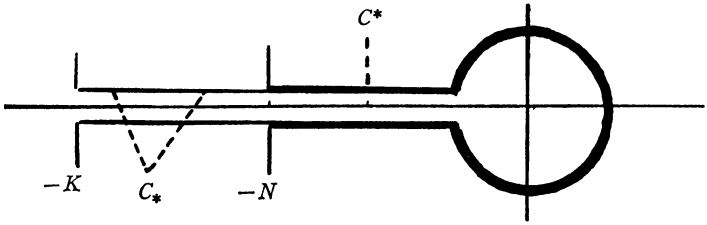
so that

$$T_1 \leqq \int_C \left|\frac{e^u}{u}\right| e^{|1-w|/|w|^{1/2}} \sum_1^\infty g(r) \frac{|1-w|^r}{2^r r!} \, |du|$$

$$\leqq A_1 \int_C \left|\frac{e^u}{u}\right| e^{|u^{-t}-1|\cdot|u|^{t/2}} \frac{|1-u^{-t}|}{1-x/2\,|1-u^{-t}|} \, |du|$$

where A_1 is a constant.

Break the path C into two parts, C^* and C_* as shown:



Given $\epsilon > 0$ choose t so small that there is a fixed positive number $N < K$ such that

$$(2.8) \qquad e^{-N} \int_{C_*} e^{|u^t-1||u|^{t/2}} \frac{|1-u^{-t}|}{1-x/2\,|1-u^{-t}|} \, |du| < \frac{\epsilon}{2}$$

and also

$$\sup_{UC^*} |1-u^{-t}| = |1-N^{-t}|$$

$$(2.9) \qquad < \frac{\epsilon}{2}\left(\int_{C^*} \left|\frac{e^u}{u}\right| e^{|u^t-1||u|^{t/2}} \frac{1}{1-x/2\,|1-u^{-t}|} \, |du|\right)^{-1}.$$

This can be done by virtue of the fact that $K \to +\infty$ as $t \to 0+$ and the fact that the integrals in (2.8) and (2.9) are seen, by elementary estimates, to be bounded independently of t. Thus $T_1 < \epsilon$, and since ϵ is arbitrary the assertion follows.

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