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A THEOREM ON FACTORIZABLE GROUPS[1]

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A group G is said to be factorizable if it contains proper subgroups H, K with the property that $G=HK$. Several results are known which state that G is not simple if H and K satisfy suitable assumptions. The following theorem and its corollaries are of a similar nature but apply only to groups of odd order. The proof is based on a theorem of H. Wielandt (see [2]) which generalized earlier results of W. Burnside and I. Schur.

THEOREM. *Let G be a group of odd order and let M be a maximal subgroup of G. Suppose that A is an abelian subgroup of G, which has at least one cyclic Sylow subgroup, such that $G=AM$. Then either G has prime order or G contains a proper normal subgroup N which is contained in either A or in M.*

PROOF. Assume that no proper normal subgroup of G is contained in M. Suppose first that $D=A\cap xMx^{-1}\neq\{1\}$, for some element x in G. Since A is abelian and since every subgroup of G conjugate to M is of the form yMy^{-1} for some element y in A, it follows that D is contained in every subgroup conjugate to M. Hence the intersection of all subgroups conjugate to M is a proper normal subgroup of G which is contained in M. This contradicts our assumption. Hence $A\cap xMx^{-1}=\{1\}$ for every element x in G.

Let π be the permutation representation of G induced by the subgroup M. As the kernel of π is contained in M, it follows from the assumptions that π is faithful. As M is a maximal subgroup of G, $\pi(G)$ is a primitive group of permutations. Since A intersects no

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conjugate of M nontrivially, the restriction of π to A is the regular representation of A. Since G has odd order, $\pi(G)$ cannot be a doubly transitive permutation group. Hence $\pi(G)$ is a permutation group which satisfies the hypotheses of Wielandt's theorem (see [**2**]). Consequently that theorem implies that A has prime order and is normal in G. The proof is completed by setting $N=A$.

COROLLARY 1. *Let G be a group of odd order which can be represented in the form $G=HA$, where H and A are proper subgroups of G, and A is an abelian group with at least one cyclic Sylow subgroup. Then G is not simple.*

PROOF. Let M be a maximal subgroup of G which contains H. Then it is easily seen that all the hypotheses of the theorem are satisfied. Thus the result follows from the theorem.

The theorem also yields a result of T. Ikuta (see [**1**]).

COROLLARY 2. *A group of odd order which contains a subgroup of prime index is either cyclic of prime order or is not simple.*

COROLLARY 3. *Let G be a group of odd order. Suppose that $G=HA$, where H is solvable and A is cyclic, then G is solvable.*

PROOF. The proof is by induction on the order g of G. The result is clearly true if G has prime order. Assume now that it has been proved for all groups of order less than g. If $G=H$ or $G=A$, the result is trivial. Hence it may be assumed that $H\neq G$, $\{1\}$. Any subgroup of G which contains H satisfies the same assumptions as G. Hence if M is a maximal subgroup of G which contains H, it follows from the induction assumption that M is solvable. The theorem now implies that G contains a normal subgroup N, such that N is contained in A or in M. In either case N is solvable. As G/N satisfies the same assumptions as G, the induction hypothesis yields that G/N is solvable. Consequently G is solvable as desired.

The assumption that G has odd order is essential in the theorem and all its corollaries, since the simple group of order 60 satisfies all the other hypotheses of the theorem and its corollaries.

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