ON THE ABSOLUTE HARMONIC SUMMABILITY OF FOURIER SERIES

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1.1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let the sequence $\{t_n\}$ be defined by

$$(1.1.1)\qquad t_n = \frac{(n+1)^{-1}s_0 + n^{-1}s_1 + \cdots + s_n}{P_n}, \qquad \left(P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right).$$

The series $\sum a_n$ is defined to be summable by harmonic means if the sequence $\{t_n\}$ tends to a limit as $n \to \infty$ [2]. If the series $\sum |t_n - t_{n-1}|$ is convergent, we say that the series is absolutely harmonic summable. It is known that this method of summability is absolutely regular and implies absolute Cesàro summability of every positive order [1].

1.2. Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Let the Fourier series associated with $f(t)$ be

$$(1.2.1)\qquad \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum u_n.$$

We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\alpha(t) = \sum_{k=0}^{\infty} (k+1)^{-1} \cos kt,$$

$$\beta(t) = \sum_{k=0}^{\infty} (k+1)^{-1} \sin kt,$$

$$\alpha_n = \int_0^{\pi} \phi(t)\alpha(t) \cos nt dt,$$

$$\beta_n = \int_0^{\pi} \phi(t)\beta(t) \sin nt dt.$$

Furthermore, A will denote an absolute positive constant and we will write $A + A = A$ and $A - A = A$.

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Zygmund [4] has shown that if $f(x)$ is of bounded variation and satisfies $|f(x+h)-f(x)| \leqq A \log^{-2-\eta}(1/h)$, $(\eta>0,\ 0\leqq x\leqq 2\pi)$, then the series (1.2.1) converges absolutely. Salem [3] has proved that this theorem is best possible in the sense that η cannot be replaced by zero.

The following theorem on the absolute harmonic summability of the series (1.2.1) is analogous to the theorem of Zygmund.

THEOREM. *If $f(x)$ is of bounded variation and satisfies*

$$(1.2.2)\qquad |f(x+h)-f(x)| \leqq A\log^{-1-\epsilon}\left(\frac{1}{h}\right), \qquad (\epsilon>0,\ 0\leqq x\leqq 2\pi),$$

then the series (1.2.1) *is absolutely harmonic summable.*

2. **Proof of the theorem.** Using (1.1.1) and (1.2.1) we have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=0}^{n-1}\left(\frac{P_k}{P_n}-\frac{P_{k-1}}{P_{n-1}}\right)u_{n-k} \\ &= \frac{1}{\pi}\int_0^\pi \phi(t)\sum_{k=0}^{n-1}\left(\frac{P_k}{P_n}-\frac{P_{k-1}}{P_{n-1}}\right)\cos(n-k)t\,dt \\ &= \frac{1}{\pi P_n P_{n-1}}\int_0^\pi \phi(t)\sum_{k=0}^{n-1}\left(\frac{P_n}{k+1}-\frac{P_k}{n+1}\right)\cos(n-k)t\,dt \\ &= \frac{1}{\pi P_n P_{n-1}}\int_0^\pi \phi(t)\left\{\sum_{k=0}^{\infty}\frac{P_n}{k+1}-\sum_{k=n}^{\infty}\frac{P_n}{k+1}-\sum_{k=0}^{n-1}\frac{P_k}{n+1}\right\} \\ &\quad\cdot\cos(n-k)t\,dt. \end{aligned}$$

Hence

$$\begin{aligned} &\pi|t_n-t_{n-1}| \\ &\leqq \frac{1}{P_{n-1}}\left|\int_0^\pi\phi(t)\left(\sum_{k=0}^{\infty}\frac{\cos(n-k)t}{k+1}\right)dt\right| \\ &\quad+\frac{1}{P_{n-1}}\left|\int_0^{1/n}\phi(t)\left(\sum_{k=n}^{\infty}\frac{\cos(n-k)t}{k+1}\right)dt\right| \\ (2.1)\quad &\quad+\frac{1}{(n+1)P_nP_{n-1}}\left|\int_0^{1/n}\phi(t)\left(\sum_{k=0}^{n-1}P_k\cos(n-k)t\right)dt\right| \\ &\quad+\frac{1}{P_{n-1}}\left|\int_{1/n}^{\pi}\phi(t)\left\{\sum_{k=n}^{\infty}\frac{\cos(n-k)t}{k+1}+\sum_{k=0}^{n-1}\frac{P_k\cos(n-k)t}{(n+1)P_n}\right\}dt\right| \\ &= \mathcal{J}_1(n)+\mathcal{J}_2(n)+\mathcal{J}_3(n)+\mathcal{J}_4(n),\ \text{say}. \end{aligned}$$

Therefore

$$(2.2)\quad \pi\sum \left| t_n - t_{n-1}\right| \leqq \sum \mathcal{g}_1(n) + \sum \mathcal{g}_2(n) + \sum \mathcal{g}_3(n) + \sum \mathcal{g}_4(n).$$

Now

$$\begin{aligned}\sum \mathcal{g}_1(n) &= \sum \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\sum_{k=0}^{\infty}\frac{\cos(n-k)t}{k+1}dt\right| \\ &= \sum \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\left\{\sum_{k=0}^{\infty}\frac{\cos nt\cos kt}{k+1} + \sum_{k=0}^{\infty}\frac{\sin nt \sin kt}{k+1}\right\}dt\right| \\ &\leqq \sum \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\alpha(t)\cos nt dt\right| \\ &\quad + \sum \frac{1}{P_{n-1}}\left|\int_0^\pi \phi(t)\beta(t)\sin nt dt\right| \qquad (2.3) \\ &= \sum \frac{|\alpha_n| + |\beta_n|}{P_{n-1}}.\end{aligned}$$

Since $1/(n+1)$ decreases steadily to zero, $\sum_{k=n}^{\infty}(\cos(n-k)t)/(k+1)$ converges for $t\neq 0$; and its sum does not exceed $4/(n+1)\left|1-e^{-it}\right|$ in absolute value, we have

$$\begin{aligned}\sum \mathcal{g}_2(n) &= \sum \frac{1}{P_{n-1}}\left|\int_0^{1/n}\phi(t)\sum_{k=n}^{\infty}\frac{\cos(n-k)t}{k+1}dt\right| \\ &\leqq A\sum\frac{1}{P_{n-1}}\left|\int_0^{1/n}\frac{1}{(n+1)t\log^{1+\epsilon}(1/t)}dt\right| \qquad (2.4) \\ &\leqq A\sum\frac{1}{(n+1)\log^{1+\epsilon} n} \leqq A,\end{aligned}$$

in view of the fact that (1.2.2) implies

$$(2.5)\qquad |\phi(t)| \leqq A\log^{-1-\epsilon}\left(\frac{1}{t}\right), \qquad (0\leqq t\leqq \pi).$$

Again, by (2.5)

$$\begin{aligned}\sum \mathcal{g}_3(n) &= \sum\frac{1}{(n+1)P_nP_{n-1}}\left|\int_0^{1/n}\phi(t)\left(\sum_{k=0}^{n-1}P_k\cos(n-k)t\right)dt\right| \\ &\leqq A\sum\frac{1}{\log n}\int_0^{1/n}\frac{1}{\log^{1+\epsilon}(1/t)}dt \qquad (2.6) \\ &\leqq A\sum\frac{1}{n\log^{2+\epsilon} n}\leqq A.\end{aligned}$$

Since $1/n(n+1)$ decreases steadily to zero,

$$\sum_{n}^{\infty} (\sin (n-k+\tfrac{1}{2})t)/k(k+1)$$

converges, and its sum does not exceed $4/n(n+1)|1-e^{-it}|$ in absolute value, we have by Abel's transformation

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{\cos (n-k)t}{k+1} &= \frac{1}{(n+1)} + \frac{1}{(n+2)}\cos t + \frac{1}{(n+3)}\cos 2t + \cdots \\ &= \frac{1}{2(n+1)} + \left(\frac{1}{(n+1)} - \frac{1}{(n+2)}\right)\frac{1}{2} \\ &\quad + \left(\frac{1}{(n+2)} - \frac{1}{(n+3)}\right)\left(\frac{1}{2} + \cos t\right) + \cdots \\ &= \frac{1}{2(n+1)} + \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \frac{\sin (n-k+1/2)t}{2\sin (t/2)} \\ &= \frac{1}{2(n+1)} + O\left(\frac{1}{n^2t^2}\right), \qquad \text{for } t \geqq \frac{1}{n}.\end{aligned} \tag{2.7}$$

Since $\sum_0^{n-1} \sin (n-k+1/2)t/(k+1) = O(1+\log(1/t))$ for $t=1/n$, we have by Abel's transformation,

$$\begin{aligned}\sum_{k=0}^{n-1} P_k \cos (n-k)t \\ &= P_0 \cos nt + P_1 \cos (n-1)t + \cdots + P_{n-1}\cos t \\ &= (P_{n-1} - P_{n-2})\left(\frac{1}{2} + \cos t\right) + \cdots \\ &\quad + (P_1 - P_0)\left(\frac{1}{2} + \cos t + \cdots + \cos (n-1)t\right) \\ &\quad + P_0\left(\frac{1}{2} + \cos t + \cdots + \cos nt\right) - \frac{1}{2}P_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{\sin (n-k+1/2)t}{2\sin (t/2)} - \frac{1}{2}P_{n-1} \\ &= O\left(\frac{1+\log (1/t)}{t}\right) - \frac{1}{2}P_{n-1}.\end{aligned} \tag{2.8}$$

Now, from (2.5), (2.7) and (2.8) we obtain

$$\sum g_4(n) = \sum \frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \phi(t) \left\{ \sum_{k=n}^{\infty} \frac{\cos (n-k)t}{k+1} + \sum_{k=0}^{n-1} \frac{P_k \cos (n-k)t}{(n+1)P_n} \right\} dt \right|$$

$$\leqq \sum \frac{A}{P_{n-1}} \int_{1/n}^{\pi} | \phi(t) | \frac{1}{n^2 t^2} dt$$

$$+ \sum \frac{A}{(n+1)\log^2 n} \int_{1/n}^{\pi} | \phi(t) | \frac{1+\log (1/t)}{t} dt$$

$$(2.9) \qquad + \sum \frac{A}{\log n} \int_{1/n}^{\pi} | \phi(t) | \left(\frac{1}{(n+1)} - \frac{P_{n-1}}{(n+1)P_n} \right) dt$$

$$\leqq A \sum \frac{1}{n^2 \log n} \int_{1/n}^{\pi} \frac{dt}{t^2 \log^{1+\epsilon} (1/t)}$$

$$+ A \sum \frac{1}{n \log^2 n} \int_{1/n}^{\pi} \frac{dt}{t \log^{\epsilon} (1/t)}$$

$$+ A \sum \frac{1}{n^2 \log^2 n} \int_{1/n}^{\pi} \frac{dt}{\log^{1+\epsilon} (1/t)}$$

$$\leqq A \sum \frac{1}{n(\log n)^{1+\epsilon}}$$

$$\leqq A.$$

Combining (2.2), (2.3), (2.4), (2.6) and (2.9) we find that, in order to prove our theorem, we need only show that

$$(2.10) \qquad \sum_{n} \frac{|\alpha_n| + |\beta_n|}{P_{n-1}} < \infty.$$

Now $\alpha(t)$ and $\beta(t)$ are continuous for $\eta \leqq t \leqq \pi$ and their absolute values when $0 < t \leqq \eta$ are each less than $A \log (1/t)$. The constants α_n and β_n are thus the Fourier coefficients of an even and odd function respectively and each of these functions belongs to L^2. The Fourier series of $\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$ is

$$-\frac{4}{\pi} \sum_{n=1}^{\infty} \alpha_n \sin nt \sin nh.$$

It follows from Parseval's theorem that

$$\sum_{n=1}^{\infty} \alpha_n^2 \sin^2 nh \leqq A \int_0^\pi \{\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)\}^2 dt$$
$$\leqq A \int_0^\pi \alpha^2(t+h) \,|\, \phi(t+h) - \phi(t-h) \,|^2 dt$$
$$+ A \int_0^\pi \phi^2(t-h) \,|\, \alpha(t+h) - \alpha(t-h) \,|^2 dt$$
$$= J_1(h) + J_2(h).$$

We first consider $J_2(h)$.

$$J_2(h) = A \int_0^\pi \phi^2(t-h) \,|\, \alpha(t+h) - \alpha(t-h) \,|^2 dt$$

(2.11)
$$\leqq A \left\{ \int_{-h}^{h} \phi^2(t)\alpha^2(t+2h)dt + \int_{-h}^{h} \phi^2(t)\alpha^2(t)dt + \int_h^\pi \phi^2(t) \,|\, \alpha(t+2h) - \alpha(t) \,|^2 dt \right\}$$
$$= J_{2,1}(h) + J_{2,2}(h) + J_{2,3}(h).$$

Using (2.5) we get

(2.12)
$$J_{2,1}(h) \leqq A \frac{1}{\log^{2+2\epsilon}(1/h)} \int_{-h}^{h} \log^2 \left(\frac{1}{t+2h} \right) dt$$
$$\leqq \frac{A}{\log^{2\epsilon}(1/h)} \int_{-h}^{h} dt \leqq \frac{Ah}{\log^{2\epsilon}(1/h)}$$

and similarly

(2.13)
$$J_{2,2}(h) \leqq A \int_0^h \phi^2(t)\alpha^2(t)dt$$
$$\leqq A \int_0^h \log^{-2-2\epsilon}\left(\frac{1}{t}\right) \cdot \log^2 \left(\frac{1}{t}\right) dt$$
$$\leqq Ah/\log^{2\epsilon}(1/h).$$

Finally,

(2.14)
$$J_{2,3}(h) \leqq Ah^2 \int_h^\pi \phi^2(t)\{\alpha'(t+2\theta h)\}^2 dt \qquad (0 < \theta < 1)$$
$$\leqq Ah^2 \int_h^\pi \log^{-2-2\epsilon}\left(\frac{1}{t}\right) \frac{dt}{(t+2\theta h)^2}$$
$$\leqq Ah/\log^{2\epsilon}(1/h).$$

Now setting $h=(\pi/2N)$ and collecting (2.11), (2.12), (2.13) and (2.14) we find that

$$J_2(h) \leqq \frac{A}{N \log^{2\epsilon} N}. \tag{2.15}$$

Since f is continuous and of bounded variation, it is clear that ϕ is also continuous and of bounded variation. Let $\omega(\delta)$ be the modulus of continuity of ϕ, and V the total variation of ϕ over $(0, 2\pi)$. We start from the inequality

$$\begin{aligned}
\sum_{k=1}^{2N} &\left\{\alpha\left(t+\frac{k\pi}{N}\right)\right\}^2 \left[\phi\left(t+\frac{k\pi}{N}\right) - \phi\left(t+(k-1)\frac{\pi}{N}\right)\right]^2 \\
&\leqq A \log^2\frac{1}{t} \sum_{k=1}^{2N}\left[\phi\left(t+k\frac{\pi}{N}\right) - \phi\left(t+(k-1)\frac{\pi}{N}\right)\right]^2 \\
&\leqq A \log^2\frac{1}{t}\, \omega\left(\frac{\pi}{N}\right) \sum_{k=1}^{2N}\left|\phi\left(t+\frac{k\pi}{N}\right) - \phi\left(t+(k-1)\frac{\pi}{N}\right)\right| \\
&\leqq A \log^2\frac{1}{t}\, \frac{V}{\log^{1+\epsilon} N},
\end{aligned}$$

which we integrate over $(0, \pi)$. On account of the periodicity, replacing x by $x+\xi$ does not affect the value of the integral, and so all integrals formed from the left hand side are equal. Hence we have,

$$\begin{aligned}
2N\int_0^{\pi} \left\{\alpha\left(t+\frac{\pi}{2N}\right)\right\}^2 &\left[\phi\left(t+\frac{\pi}{2N}\right) - \phi\left(t-\frac{\pi}{2N}\right)\right]^2 dt \\
&\leqq \frac{A}{\log^{1+\epsilon} N}\int_0^{\pi} \log^2\frac{1}{t}\, dt \\
&\leqq \frac{A}{\log^{1+\epsilon} N}.
\end{aligned}$$

Hence

$$J_1\left(\frac{\pi}{2N}\right) \leqq \frac{A}{N \log^{1+\epsilon} N}. \tag{2.16}$$

Combining (2.15) and (2.16) we find that

$$\sum_{n=1}^{\infty} \alpha_n^2 \sin^2\left(\frac{n\pi}{2N}\right) \leqq \frac{A}{N \log^{2\epsilon} N} \qquad (\epsilon < 1).$$

Taking $N=2^{\nu}$, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \alpha_n^2 \leqq 2 \sum_{n=2^{\nu-1}+1}^{2^\nu} \alpha_n^2 \sin^2\left(\frac{n\pi}{2^{\nu+1}}\right)$$
$$\leqq 2 \sum_{n=1}^{\infty} \alpha_n^2 \sin^2\left(\frac{n\pi}{2^{\nu+1}}\right)$$
$$\leqq A(2^{-\nu}\cdot\nu^{-2\epsilon}).$$

Therefore, applying Schwarz's inequality we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |\alpha_n| \log^{-1} n \leqq \left\{\sum_{n=2^{\nu-1}+1}^{2^\nu} \alpha_n^2\right\}^{1/2} \left\{\sum_{n=2^{\nu-1}+1}^{2^\nu} \log^{-2} n\right\}^{1/2}$$
$$\leqq A\left\{2^{-\nu/2}\cdot\nu^{-\epsilon}\right\} 2^{(\nu-1)/2}\cdot(\nu-1)^{-1}$$
$$\leqq A/\nu^{1+\epsilon}.$$

A similar relation holds in the case of β_n. Hence

$$\sum_{n=1}^{\infty} \frac{|\alpha_n| + |\beta_n|}{P_{n-1}} \leqq A \sum_{n=1}^{\infty} \frac{|\alpha_n|}{\log n} + A \sum_{n=1}^{\infty} \frac{|\beta_n|}{\log n}$$
$$= A \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|\alpha_n|}{\log n} + A \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|\beta_n|}{\log n}$$
$$\leqq A \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\epsilon}} \leqq A;$$

and thus the proof of the theorem is complete.

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