A VARIANT OF HELLY'S THEOREM[1]

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1. Helly's [8] theorem on intersections of convex sets ("If every $k+1$ members of a family of compact, convex subsets of E^k have a nonempty intersection, then the intersection of all the members of the family is not empty") has been generalized in various directions. Helly himself gave (in [9]) a generalization to families of not necessarily convex sets, in which the intersections of any $2, 3, \cdots, k$ members are assumed to satisfy certain conditions (which are automatically fulfilled for families of convex sets). In other papers (e.g., [3; 7; 13]) problems related to Helly's theorem were considered under weaker assumptions on the number of sets with nonempty intersections, but restricted to families consisting of translates (or of homothetic images) of one convex set. Similar families have been considered also in connection with theorems of Helly's type for common transversals instead of common points (a list of references is given in [4]). On the other hand, it was shown [2] that if families of affine transforms of one set are considered, the convexity of the sets is necessary for the validity of Helly's theorem in its original form.

The present paper results from an attempt to find whether there exists some theorem of Helly's type for nonconvex sets, in case only families consisting of affine or other appropriate transforms of one set are considered and, possibly, additional conditions imposed. But, as is easily seen by a slight modification of the example on p. 70 of [5] (or by Example 1 of the present paper), even if only families of translates are considered, and the sets assumed to consist of only two convex components, there exists no "critical number" corresponding to $k+1$ in Helly's theorem.

Nevertheless, the following "individual" theorem is valid and, as shown by the examples in §4, in many respects the best possible.

THEOREM 1. *Let $K \subset E^k$ be the union of a finite number of disjoint, compact, convex sets. There exists an integer $n = n(K)$ with the following property: If $\mathcal{K} = \{K_\alpha : \alpha \in A\}$ is any family of subsets of a real vector space E, such that each K_α is the transform of K by a nonsingular affine*

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transformation T_α (possibly depending on $\alpha \in A$), and such that any n members of $\mathcal{K}$ have a nonempty intersection, then the intersection of all the members of $\mathcal{K}$ is not empty.

We shall obtain Theorem 1 as a corollary of a more general result (Theorem 2 in §3). In §2 some Lemmata are proved which are needed to establish Theorem 2; examples showing the necessity of some of the conditions are given in §4.

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2. We start with some definitions and results which are used in §3.

DEFINITION 1. *Let S be a compact, convex subset of E^k; for any real $\delta > 0$ let $M(S, \delta)$ denote the minimal number of translates of $\delta S = \{x;\ x = \delta y,\ y \in S\}$ which may cover S.*

With this notation we have

LEMMA 1. *For any compact, convex $S \subset E^k$ and any $\delta > 0$ the inequality $M(S, \delta) \leqq [(k^2+\delta)/\delta]^k$ holds. If S has, moreover, a center of symmetry, then $M(S, \delta) \leqq [(k+\delta)/\delta]^k$.*

PROOF. Assuming that the origin 0 is the centroid of S, let $S^* = [S+(-S)]/2$ ("Zentralsymmetrisierung"). It has been shown (see, e.g., [**6**; **10**; **12**]) that $S \subset (2k/(k+1))S^* \subset kS$. Therefore, the first inequality of Lemma 1 is a consequence of the second one. We shall establish the second inequality as follows: It is well-known [**1**; **14**] that for any centrally symmetric $S^* \subset E^k$ there exists a parallelohedron P containing S^* and such that the centers of the $(k-1)$ dimensional faces of P belong to S^*. Therefore, if 0 is the center of S^*, we have $P/k \subset S^*$. Since P may obviously be covered by $([k/\delta]+1)^k \leqq [(k+\delta)/\delta]^k$ translates of $(\delta/k)P$, and since $(\delta/k)P \subset \delta S^*$ while $S^* \subset P$, it follows that S^* may be covered by $\leqq [(k+\delta)/\delta]^k$ translates of δS^*. This ends the proof of Lemma 1.

REMARK 1. The estimate of Lemma 1 seems to be very crude, although it is sufficient for our purposes. In case $k=2$ the slightly better bounds $[(2+\delta)/\delta]^2$ and $[(3/2+\delta)/\delta]^2$ may be obtained by more elaborate arguments.

The next Lemma will be used later only in the particular case $r=1$, $q=k+2$.

LEMMA 2. *For any positive integers k, q, r and any real δ, $0 < \delta \leqq 1$ there exists an integer $N = N(k, q, r, \delta)$ with the following property:*

Given any set $X \subset E^k$ containing at least N points, it is possible to find a subset X_0 of X, which contains at least q points and satisfies the condi-

tion: If Y is a subset of X obtained from X by omitting at most r points, and if $C=\mathcal{C}(Y)$ is the convex hull of Y, there exists a translate of δC covering $X_0\cap Y$.

Moreover, we may take $N(k, q, r, \delta)=[q(k+1)^{3k}\delta^{-k}]+(r+1)(k+1)^2$.

PROOF. If $k=1$ it is easily seen that the Lemma holds already if X contains $r+2q\delta^{-1}\leqq N(1, q, r, \delta)$ points. We proceed by induction, assuming $k>1$ and the Lemma proved for E^m, $m<k$. Let Z_j, for $0\leqq j\leqq r$, denote subsets of X consisting of $k+1$ points each, determined as follows: Z_0 consists of such points of X for which the volume of their convex hull $T_0=\mathcal{C}(Z_0)$ is maximal among all simplices with vertices belonging to X. For $i\geqq 1$ let Z_i be a $(k+1)$-tuple of points of $X\sim(\bigcup_{j<i} Z_j)$ (set-theoretic difference) which maximizes the volume of the simplex $T_i=\mathcal{C}(Z_i)$. If the volume of T_r is 0 the Lemma is proved, since then the set $X'=X\sim(\bigcup_{0\leqq i\leqq r} Z_i)$ is of dimension at most $(k-1)$ and $N(k, q, r, \delta)\geqq(r+1)(k+1)+N(k-1, q, r, \delta)$. Thus we may assume that all the simplices T_i are nondegenerate.

The maximality of the volume of T_i implies that, for each i with $0\leqq i\leqq r$, the set X' is contained in a suitable translate of $-kT_i$. It follows that X' may be covered by translates of $2kT_i^*=k[T_i+(-T_i)]$ having their centers at any chosen point of X'. Therefore X' is contained in a suitable translate of $2k(\bigcap_{0\leqq i\leqq r} T_i^*)=2kT^*$. Using the notation of Lemma 1, let $m=M(T^*, \delta/k(k+1))\leqq[(k^2(k+1)+\delta)/\delta]^k$. Then X', being contained in a translate of $2kT^*$, may be covered by m translates of $(2\delta/(k+1))T^*$. Since $N(k, q, r, \delta)\geqq(r+1)(k+1)+qm$, at least one of these m translates, say $x_0+(2\delta/(k+1))T^*$, contains q or more points. We put $X_0=(x_0+(2\delta/(k+1))T^*)\cap X'$. On the other hand, any possible Y contains at least one of the sets Z_i. Therefore $C=\mathcal{C}(Y)$ contains the corresponding T_i and thus (as in the proof of Lemma 1) also a translate of $(2/(k+1))T_i^*$. A fortiori, C contains a translate of $(2/(k+1))T^*$, and δC one of $(2\delta/(k+1))T^*$. But then a suitable translate of δC covers X_0. This ends the proof of Lemma 2.

REMARK 2. Obviously the estimate of $N(k, q, r, \delta)$ given above is very crude; it could be somewhat sharpened even by using essentially the same method, at the cost of more involved calculations.

DEFINITION 2. *For a compact subset K of E^k we define a "measure of nonconvexity" $\Delta(K)$ (where $0\leqq\Delta(K)\leqq\infty$) as follows:*

For any straight line L in E^k, let $D_L(K)$ denote the maximum of lengths of segments parallel to L and contained in $\mathcal{C}(K)$; let $d_L(K)$ denote the greatest lower bound of the lengths of the (open) intervals constituting $L\cap(E^k\sim K)$. Then $\Delta(K)=\sup_L D_L(K)/d_L(K)$, where the

greatest lower bound is taken over all lines L in E^k. (If $d_L(K)=0$ for some L with $D_L(K)>0$, we put $D_L(K)/d_L(K)=\infty$.)

Obviously, $\Delta(K)=0$ if and only if K is convex.

LEMMA 3. *If K is compact then $\Delta(K)<\infty$ if and only if K is the union of a finite number of disjoint, compact, convex sets.*

PROOF. The "if" part being obvious, we proceed to prove the "only if" part of the Lemma. We remark, first, that $\Delta(K)<\infty$ implies, by the compactness of K, that the number of connected components of K must be finite. Next, if K^* is any connected component of K then K^* is compact and $\Delta(K^*)<\infty$. The convexity of K^* now follows from Tietze's theorem [**11**; **15**], according to which a set is convex if it is compact, connected and locally convex. Indeed, K^* must be locally convex since otherwise there would exist arbitrarily small segments with end-points in K^* which are not contained in K^*, and therefore we would have $\inf_L d_L(K^*)=0$, while $D_L(K^*)$ may be assumed to be bounded away from 0, thus contradicting $\Delta(K^*)<\infty$. This ends the proof of Lemma 3.

3. Now we may prove

THEOREM 2. *For any ϵ, $0\leqq\epsilon<\infty$, and any natural k there exists an $n=n(k,\epsilon)$ with the following property:*

If $\mathcal{K}=\{K_\alpha;\alpha\in A\}$ is any family of compact subsets of E^k such that: (i) $\Delta(K_\alpha)\leqq\epsilon$ *for all $\alpha\in A$;* (ii) *any n members of $\mathcal{K}$ have a nonempty intersection; then $\bigcap_{\alpha\in A}K_\alpha\neq\emptyset$.*

Moreover, we may take $n(k,\epsilon)=[(k+2)(k+1)^{3k}\epsilon_0^k]+2(k+1)^2$, where $\epsilon_0=\max(\epsilon,1)$.

PROOF. Assuming the Theorem false, let $\mathcal{K}$ be some family contradicting its assertion. Since the members of $\mathcal{K}$ are compact, we may obviously assume that the index-set A is finite, and contains at least $n(k,\epsilon)$ elements. Moreover, it may be assumed that $\mathcal{K}$ is minimal, i.e. that for each $\alpha\in A$ we have $K^\alpha=\bigcap_{\beta\in A,\beta\neq\alpha}K_\beta\neq\emptyset$ although $\bigcap_{\alpha\in A}K_\alpha=\emptyset$.

For each $\alpha\in A$ let $x_\alpha\in K^\alpha$. To the set $X=\{x_\alpha;\alpha\in A\}$ we apply Lemma 2, with $q=k+2$, $r=1$ and $\delta<1/\epsilon_0$. Let X_0 be the subset of X whose existence was established in Lemma 2. According to Lemma 3, each $K_\alpha\in\mathcal{K}$ is the union of a finite number of compact, convex, mutually disjoint sets $K_\alpha^{(i)}$, $1\leqq i\leqq p_\alpha$. We claim that for each $\alpha\in A$ there exists a component $K_\alpha^{(i_\alpha)}$ of K_α such that $X_0\cap K_\alpha=X_0\cap K_\alpha^{(i_\alpha)}$. Indeed, using the same notation as in the proof of Lemma 2, K_α contains either Z_0 or Z_1 (or both), and therefore the convex hull C_α of K_α

contains at least one of T_0 and T_1. Since $X_0 \subset (2\delta/(k+1))T_0$, the distance of any two points x, $x^* \in X_0 \cap K_\alpha \subset X_0$ is $\leqq \delta$ times the length of some segment (parallel to that determined by x and x^*) contained in a translate of $(2/(k+1))T^*$ which is contained in C_α. Therefore, the assumption that x and x^* belong to different components of K_α contradicts the condition $\Delta(K_\alpha) \leqq \epsilon$.

Now, $X_0 \cap K_\alpha^{(t_\alpha)}$ is either equal to X_0, or else is obtained from X_0 by omitting one point (x_α, if $x_\alpha \in X_0$). Since X_0 contains at least $k+2$ points, any $k+1$ of the sets $X_0 \cap K_\alpha^{(t_\alpha)}$ have a nonempty intersection. A fortiori, the intersection of any $k+1$ of the sets $K_\alpha^{(t_\alpha)}$ is not empty. It follows then from Helly's theorem that $\emptyset \neq \bigcap_{\alpha \in A} K_\alpha^{(t_\alpha)} \subset \bigcap_{\alpha \in A} K_\alpha$. But this is in contradiction to our assumption on the family $\mathcal{K}$. Thus Theorem 2 is proved.

Theorem 1 is now readily established. Indeed, let ϵ satisfy $\infty > \epsilon \geqq \Delta(K)$ and let $n(K) = n(k, \epsilon) + 1$. We take a fixed $\alpha_0 \in A$. If $\mathcal{K}^* = \{K_\alpha^* = K_\alpha \cap T_{\alpha_0}(E^k);\ \alpha \in A\}$ then any $n(k, \epsilon)$ members of $\mathcal{K}^*$ have a nonempty intersection. On the other hand, since $\Delta(K)$ is invariant under nonsingular affine transformations of K, and since for any compact set S and any linear variety V with $S \cap V \neq \emptyset$ obviously $\Delta(S \cap V) \leqq \Delta(S)$, it follows that $\Delta(K_\alpha^*) \leqq \epsilon$ for each $\alpha \in A$. But then, by Theorem 2, $\bigcap_{\alpha \in A} K_\alpha^* \neq \emptyset$, which establishes Theorem 1.

4. The assumptions of Theorem 1 cannot be omitted, or its conclusion strengthened, as is shown by the following examples. Except for Example 5, the transformations T_α may even be restricted to translations.

We first show that $\sup_K n(K) = \infty$, even if only one-dimensional sets K are considered which, moreover, consist either of only two convex components (segments) or else are finite.

EXAMPLE 1. Let G_m be the union of the two segments $[0, (m-1)/2m]$ and $[(m+1)/2m, 1]$. Then $n(G_m) > m$.

EXAMPLE 2. Let F_m be the set of $2m$ points (on the real line) with coordinates $k/2m$, where $0 \leqq k \leqq 2m$ but $k \neq m$. Then obviously $n(F_m) > m$.

If K is the union of two convex sets which have a nonempty intersection, it is possible to have $n(K) = \infty$.

EXAMPLE 3. In E^2, let K_1 be the convex hull of the points $(0, 0)$, $(-1, 0)$, $(-1, 1)$, and let K_2 be the convex hull of $(0, 0)$, $(1, 0)$, $(1, 1)$. Then it is easy to find, for any natural m, families of translates of $K = K_1 \cup K_2$ which show that $n(K) > m$.

Theorem 1 may fail if the number of components of K is infinite, even if each of them is convex and has a nonempty interior, as shown by the following example.

EXAMPLE 4. Let K be the following union of (closed) segments: $K=[-1, 0]\cup(\bigcup_{k=1}^{\infty}[1/(k+1)+1/(k+1)^3, 1/k])\cup[1, 2]$. There exist intersections of three suitable translates of K similar to the set G_m of Example 1, for any m. Therefore, $n(K)=\infty$.

On the other hand, Theorem 1 may fail also if the affine transformations T_α are not assumed to be nonsingular.

EXAMPLE 5. If $K\subset E^2$ is the union of two disjoint, congruent circular discs, singular affine transforms of K may be similar to G_m, for any sufficiently great m.

Added in proof. A paper *On components in some families of sets* is being prepared by T. S. Motzkin and the present author; in it theorems related to Helly's are obtained in a more general setting.

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