ON GROUPS OF DIFFEOMORPHISMS

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I. We consider here the groups of homeomorphisms on Euclidean n-space and the n-sphere S^n. Chiefly we will be concerned with the question of whether or not these groups reduce in an homotopy sense to the ordinary orthogonal group acting on these spaces. Such questions are intimately connected with the theory of fibre bundles in which these spaces occur as fibres. We will restrict ourselves to the case where the homeomorphisms are of class C^1 and will topologize the various groups taking account of the differentiability.

We first consider Euclidean n-space E^n. We denote by K the group of all homeomorphisms f of E^n such that f and f^{-1} are of class C'. K becomes a topological group by demanding uniform convergence of f and its derivatives on compact sets,[1] i.e. a typical neighborhood of the identity function is given by

$$(1) \quad V_{r,\epsilon} = \left\{ g \in K \,\Big|\, \|g(x) - x\| < \epsilon, \left| \frac{\partial g_i}{\partial x_k}(x) - \delta_k^i \right| < \epsilon, \|x\| < r, \right.$$

$$\left. i, k = 1, \cdots, n \right\}.$$

Clearly the orthogonal group, O_n, of E^n is imbedded in K and the topology induced on O_n is the usual topology. In fact any locally compact or complete metric group which acts as a transformation group of E^n so that each motion is of class C^1 is imbedded in K (see [1, p. 197]). With this topology on K we now show

THEOREM I. *O_n is a deformation retract of K.*

First we will demonstrate a number of lemmas which lead to the proof of Theorem I.

LEMMA 1. *Let $K_0 = \{g \in K \mid g(0) = 0\}$. Then K_0 is a closed subgroup of K and K decomposes topologically into $E^n \times K_0$. Hence if O_n is a deformation retract of K_0, then O_n is a deformation retract of K.*

PROOF. With each $x \in E^n$ associate $\sigma_x \in K$, $\sigma_x(y) = y + x$. Clearly $x \rightarrow \sigma_x$ is topological. If $f \in K$, consider $f(0) = x_0$. Then $\sigma_{-x_0} f$ is in K_0

[1] For details on this topology, see R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble vol. 6 (1956) pp. 43–87.

and $f = \sigma_x g$, $g \in K_0$. Furthermore, the decomposition is unique. Indeed if $\sigma_x g = \sigma_y h$ then $gh^{-1} = \sigma_{y-x}$ is a translation. Since $gh^{-1} \in K_0$, $y = x$, $g = h$. That the topology on K is the topology of the product space is easily established.

Now let

$$H = \left\{ g \in K_0 \left| \frac{\partial g_i}{\partial x_k} = \delta_k^i \right. \right\}.$$

H is then a closed, invariant subgroup and

LEMMA 2. K_0 *decomposes topologically into* $\mathrm{Gl}(n, R) \times H$ *where* $\mathrm{Gl}(n, R)$ *is the group of non-singular matrices on* E^n.

PROOF. Consider the homomorphism $\phi: K_0 \to \mathrm{Gl}(n, R)$ defined by $\phi(f) = ((\partial f_i / \partial x_k)(0))$. ϕ is continuous on K_0 since we have demanded close partial derivatives. If $\alpha \in \mathrm{Gl}(n, R)$ then $\phi(\alpha) = \alpha$. If $f \in K_0$ consider the element $\phi(f)^{-1} f$ under ϕ. We have

$$\phi(\phi(f)^{-1} f) = \phi(\phi(f)^{-1}) \phi(f) = \phi(f)^{-1} \phi(f) = I$$

the identity matrix, and $\phi(f)^{-1} f$ is in the kernel of ϕ which is clearly H, and $f = \phi(f) g$, $g \in H$. If $\alpha g = \beta h$, α, $\beta \in \mathrm{Gl}(n, R)$, g, $h \in H$ then $\beta^{-1} \alpha = h g^{-1}$ and since $h g^{-1} \in H$, $\beta^{-1} \alpha = I$ and consequently $\alpha = \beta$ and $g = h$. Again the product topology of K_0 can be shown to be the product topology on $\mathrm{Gl}(n, R)$ and H.

LEMMA 3. *The group* H *is contractible to a point.*

PROOF. For $0 < t < \infty$, denote by L_t the transformation $L_t x = tx$. Then we define the homotopy $\Phi: H \times [0, 1] \to H$ by

$$\Phi(f, t) = \begin{cases} L_{1/t} f L_t, & 0 < t \leq 1, \\ I, & t = 0. \end{cases}$$

We have $\Phi(f, 1) = f$ and $\Phi(f, 0) = I$, thus if Φ is continuous, Φ contracts H to I. First we see that Φ is continuous on $H \times [0, 1]$ since on this set Φ amounts to a continuous system of inner automorphisms and these are continuous since the group operations are continuous.

Now let $f \in H$, we will show that Φ is continuous at $(f, 0)$. Suppose we are given a neighborhood $V_{r,\epsilon}$ of I as given by (1), (we suppose $r > 1$). We must show that there exists a neighborhood, U, of f in H and a $\delta > 0$ such that for $g \in U$, $t < \delta$, $\Phi(g, t) \in V_{r,\epsilon}$. Now if $h \in H$ then by the theorem of the mean we have

$$h_i(x) = x_i + \sum_{k=1}^{n} c_i^k(h; x) x_k$$

and $c_i^k(h; x)$ tends to 0 with x, (in fact $c_i^k(h; x) = (\partial h_i/\partial x_k)(\theta_i^t x)$ $- \delta_i^k$, $0 < \theta_i^t < 1$). Furthermore, denoting by h^t the transformation $\Phi(h, t)$, $t > 0$ we have $(\partial h_i^t/\partial x_j)(x) = (\partial h_i/\partial x_j)(tx)$. Then we first select a neighborhood U_1 of f so that for

$$g \in U_1, \quad \|x\| \leq r, \quad \left| \frac{\partial g_i}{\partial x_k}(x) - \frac{\partial f_i}{\partial x_k}(x) \right| < \frac{\epsilon}{2rn}.$$

Since $(\partial f_i/\partial x_k)(0) = \delta_k^i$ we can choose $\eta > 0$ so that $\|x\| < \eta$ implies $\left| (\partial f_i/\partial x_k)(x) - \delta_k^i \right| < \epsilon/2rn$. Choose $\delta_1 > 0$ so that for $0 < t < \delta_1$, $\|x\| \leq r$ we have $\|tx\| < \eta$. Under these conditions we obtain for $g \in U_1$, $0 < t < \delta_1$, $\|x\| \leq r$

$$\left| \frac{\partial \overset{t}{g_i}}{\partial x_k}(x) - \delta_k^i \right| = \left| \frac{\partial g_i}{\partial x_k}(tx) - \delta_k^i \right|$$

$$\leq \left| \frac{\partial g_i}{\partial x_k}(tx) - \frac{\partial f_i}{\partial x_k}(tx) \right| + \left| \frac{\partial f_i}{\partial x_k}(tx) - \delta_k^i \right|$$

$$< \frac{\epsilon}{rn}.$$

Since the inequality holds for all x for which $\|x\| \leq r$ it also holds for θx, $0 < \theta < 1$, $\|x\| \leq r$ and we conclude that for $g \in U_1$, $\left| c_i^k(g; tx) \right|$ $< \epsilon/rn$. Then

$$\left| \overset{t}{g_i}(x) - x_i \right| = \left| \sum_{j=1}^n \overset{j}{c_i}(g; tx)x_j \right|$$

$$\leq \sum_{j=1}^n \left| \overset{j}{c_i}(g; tx) \right| \left| x_j \right|$$

$$< \epsilon.$$

Hence Φ maps $U_1 \times [0, \delta_1)$ into $V_{r,\epsilon}$ and Φ is continuous at $(f, 0)$ and the lemma is proved.

PROOF OF THEOREM I. It follows from Lemmas 1, 2, 3 that $K = E^n \times Gl(n, R) \times H$ as a topological space. Now $Gl(n, R)$ can be topologically decomposed as the product of the orthogonal group and a euclidean space. Then K is topologically the product $O_n \times A$ where A is a contractible space, and Theorem I follows readily.

COROLLARY I. *Let $\mathfrak{B} = \{B, p, X, E^n, K\}$ be a fibre bundle in the sense of Steenrod, [2, p. 8], with fibre E^n, group K and base space X a locally finite polyhedra. Then $\mathfrak{B}$ is equivalent in K to a bundle with group O_n.*

PROOF. According to [2, p. 36] it suffices to show that the bundle $\mathcal{B}'$ associated to $\mathcal{B}$ with fibre K/O_n has a cross-section. It follows from Theorem I that $\Pi_i(K/O_n) = 0$, $i \geq 0$. Since X is a locally finite polyhedra it follows in the usual way that we can define a cross-section inductively on the k-dimensional skeleton of X and obtain the corollary.

II. We consider now the sphere, S^n, and the group K of homeomorphisms f of S^n such that f and f^{-1} are of class C^1 in the usual differentiable structure of S^n. We suppose S^n covered by two coordinate neighborhoods, U_1 and U_2, $U_1 = S^n \setminus \{x_0\}$, $U_2 = S^n \setminus \{-x_0\}$. If (x) are the coordinates on U_1 and (y) the coordinates on U_2 we take as coordinate transformations

$$y_i = \frac{x_i}{\sum\limits_{k=1}^{n} x_k^2} .$$

K is topologized with the usual C^1-topology (cf. the reference in footnote 1).

THEOREM II. *If O_{n+1} is the orthogonal group then $i_*: \Pi_k(O_{n+1}) \to \Pi_k(K)$ is an isomorphism into where i_* is induced by the injection $i: O_{n+1} \to K$.*

The proof follows

LEMMA 4. *Let K_0 denote the subgroup of K which holds x_0 fixed. Then $i: O_n \to K_0$ induces an isomorphism i_* of $\Pi_k(O_n)$ into $\Pi_k(K_0)$ for each k.[2]*

PROOF. We suppose x_0 to have coordinates $(0, \cdots, 0)$ in U_1. Define the continuous homomorphism $\phi: K_0 \to \mathrm{Gl}(n, R)$ by

$$\phi(g) = \left(\frac{\partial g_i}{\partial x_k}(0) \right).$$

For each g, $\phi(g)$ can be expressed uniquely and continuously as the product of an orthogonal matrix $\psi(g)$ and a triangular matrix $\pi(g)$, $\phi(g) = \psi(g)\pi(g)$. Then $\psi: K_0 \to O_n$ and for $h \in O_n$, $\psi(h) = h$ so that ψ is a retraction of K_0 onto O_n. The lemma then follows immediately.

PROOF OF THEOREM II. It is easily seen that the map $f: K \to S_n$, $f(g) = g(x_0)$ is open (since f is open on O_{n+1}) and consequently that K/K_0 is canonically homeomorphic to S^n. Furthermore, the local cross-section of O_{n+1} over S^n provides a local cross-section for K over

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[2] O_n can be imbedded in K_0 so that its action is that of O_n on the coordinate neighborhood U_1.

S^n, and K is then a fibre bundle over S^n. Consequently we have the following commutative diagram of exact sequences

$$\cdots \to \Pi_k(O_n) \to \Pi_k(O_{n+1}) \to \Pi_k(S^n) \to \Pi_{k-1}(O_n) \to \Pi_{k-1}(O_{n+1}) \to \cdots$$
$$\downarrow i_* \qquad \downarrow i_* \qquad \downarrow j_* \qquad \downarrow i_* \qquad \downarrow i_*$$
$$\cdots \to \Pi_k(K_0) \to \Pi_k(K) \to \Pi_k(S^n) \to \Pi_{k-1}(K_0) \to \Pi_{k-1}(K) \to \cdots$$

where j_* is the identity. It is easily seen that $\mathrm{Ker}(i_* : \Pi_k(O_{n+1}) \to \Pi_k(K))$ is the image under $\Pi_k(O_n) \to \Pi_k(O_{n+1})$ of $\mathrm{Ker}(i_* : \Pi_k(O_n) \to \Pi_k(K_0))$. Since this last kernel is O the theorem follows.

REFERENCES

1. D. Montgomery and L. Zippin, *Topological transformation groups*, New York, Interscience Publishers, 1955.

2. N. Steenrod, *The topology of fibre bundles*, Princeton, Princeton University Press, 1951.

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A NOTE ON GAUSS' FIRST PROOF OF THE QUADRATIC RECIPROCITY THEOREM

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We assume that the reader is familiar with Mathews' exposition [1, pp. 45–50] of the inductive proof of the reciprocity theorem. There are three main cases:

I. pRq,

II. pNq, $q \equiv 3 \pmod{4}$,

III. pNq, $q \equiv 1 \pmod{4}$.

In I we have $e^2 - p = qf$, in II we have $e^2 + p = qf$. In III we have first the lemma which asserts the existence of a prime $p' < q$ such that qNp'. This implies $p'Nq$, so that $pp'Rq$ and so $e^2 - pp' = qf$. In each of the cases I and II it is necessary to treat two sub-cases; in case III there are four sub-cases. Thus in all there are eight cases to consider.

We should like to point out in this note that it is possible to handle all cases simultaneously by introducing a little notation. To begin with, we define