METRIZABLE KÖTHE SPACES[1]

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1. Let E be a locally compact Hausdorff space, μ a Radon measure, [3], on E, and $\Omega(E, \mu)$ the space of equivalence classes of locally integrable functions on E with respect to the measure μ. Here, two functions f and g are equivalent if $f-g=0$ except on a set S which meets every compact set in a set of measure zero; the notation $f=\lim f_n$ applies to equivalence classes. For a subset Γ of Ω, let

$$\Lambda(\Gamma) = \left\{f \in \Omega: \left|\int fg d\mu\right| < \infty \text{ for all } g \in \Gamma\right\}.$$

The sets $\Lambda=\Lambda(\Gamma)$ and $\Lambda^*=\Lambda(\Lambda)$ are vector lattices, [1], and are called associated Köthe spaces. Initially Köthe and Toeplitz, [9], and later Köthe, in a series of papers of which [10] is representative, studied these spaces for the case where E is the space of natural numbers with the discrete topology and $\mu(n)=1$ for every natural number n. Dieudonné, [4], extended the theory to the case for which E is σ-compact. Köthe spaces, which are also Banach spaces, were studied by Lorentz and Wertheim [7] for the case where $E=[0, 1]$ and μ is Lebesgue measure. Examples of Köthe spaces are the Lebesgue spaces L_p, the Orlicz spaces L_Φ, and arbitrary intersections of such spaces.

For each Köthe space Λ, the associated space Λ^* determines a family of topologies in Λ. These topologies are locally convex, Hausdorff, and are compatible with the order relation in Λ. Among them there is a weakest and a strongest. It should be observed that the strongest of these topologies may be strictly stronger than the Mackey topology $m(\Lambda, \Lambda^*)$, [2]. In the case that E is σ-compact Dieudonné [4] showed that the Köthe space Λ is complete for each of these topologies. Later, Goffman [5], using the work of Nakano, observed that the restriction of σ-compactness is not necessary.

In this paper a characterization is obtained of those Köthe spaces which, with their strongest Köthe topologies $S(\Lambda, \Lambda^*)$, are Banach spaces. A slight modification of these conditions gives a characterization of those Köthe spaces Λ, which, with their strongest Köthe topologies $S(\Lambda, \Lambda^*)$, are Fréchet spaces. It is shown that Λ with the

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topology $S(\Lambda, \Lambda^*)$ is a Banach space if and only if Λ^* is a Banach space with the topology $S(\Lambda^*, \Lambda)$. If Λ with the topology $S(\Lambda, \Lambda^*)$ is a Fréchet space, but not a Banach space, then Λ^* is not metrizable for the topology $S(\Lambda^*, \Lambda)$. As Λ^* is not in general the topological dual of Λ with the topology $S(\Lambda, \Lambda^*)$ these results do not follow from the standard theory. In general the Köthe space Λ has more than one Köthe topology. In §5 is considered a case where this is not so.

2. Let W be a family of weakly bounded subsets of Λ^* whose union is all of Λ^*. The weak duality determined for Λ and Λ^* by the bilinear form

$$(f, g) = \int fg d\mu,$$

$f \in \Lambda$, $g \in \Lambda^*$ makes it possible to define for Λ the topology of uniform convergence on sets in W. With this topology Λ is a locally convex space. If the sets in W are normal[2] as well as weakly bounded then this topology is called a Köthe topology and is denoted by $\mathcal{K}_w(\Lambda, \Lambda^*)$. The strongest and weakest such topologies are determined when W contains all weakly bounded normal subsets of Λ^* and W contains only the normal closures[3] of points in Λ^* respectively. The weak topology $\sigma(\Lambda, \Lambda^*)$ is in general weaker than the weakest Köthe topology.

3. For a nonempty class C of positive integrable functions $c(t)$ on $[0, 1]$, Lorentz and Wertheim [7], define the Köthe space $X(C)$ as the set of measurable functions $f(t)$ for which

$$\|f\| = \sup_{c \in C} \int_0^1 | f(t) | c(t) dt < \infty.$$

They show that if certain additional assumptions are placed on the set C, then their definition is equivalent to that given by Dieudonné, and their space $X(C)$ is a Banach lattice. Their additional conditions give the motivation for the following definition.

A nonempty subset A of the Köthe space Λ is said to satisfy condition (*), if for each nondecreasing sequence (f_n) of nonnegative functions in A, there exists a function f in A such that $f \geqq f_n$ for $n = 1, 2, \cdots$.

[2] A subset A of partially ordered vector space X is normal if $x \in X$, $y \in A$, and $|x| \leqq |y|$ implies $x \in A$.

[3] The P closure, of a subset A of Λ, is the smallest subset of Λ containing A which is P.

Normal convex subsets A of Λ which satisfy condition (*) will occur so often in the discussion that they will be referred to as admissible sets. Admissible sets play a fundamental role for:

THEOREM 1. *Every admissible subset of the Köthe space* Λ *is weakly bounded.*

PROOF. Suppose that A is an admissible subset of Λ which is not weakly bounded. There exists a function g in Λ^*, and a sequence (f_n) in A such that

$$\left| \int f_n g d\mu \right| > 2^n, \qquad n = 1, 2, \cdots .$$

Let $h_n = \sum_{i=1}^{n} 2^{-i}|f_i|$. Because $h_{n+1} \geqq h_n \geqq 0$ for each integer n, and because A is admissible there exists a function h in A such that $h \geqq h_n$ for $n = 1, 2, \cdots$. Now

$$\int |hg| \, d\mu \geqq \int |h_n g| \, d\mu > n, \qquad n = 1, 2, \cdots .$$

This implies that $|hg|$ is not integrable, while hg is integrable. Hence A is weakly bounded.

For a subset A of the Köthe space Λ,

$$A^0 = \left\{ g \in \Lambda^* : \left| \int fg d\mu \right| \leqq 1 \text{ for all } f \in A \right\}.$$

If A is an absorbing subset of Λ, then A^0 is a weakly bounded subset of Λ^*. If A is a weakly bounded subset of Λ, then A^0 is an absorbing subset of Λ^*. For proofs of these facts together with the fact that $(A^0)^0 = A^{00}$ is the weak convex closure of A, see [2].

A consequence of the following lemma is that every Köthe topology $\mathcal{K}_w(\Lambda, \Lambda^*)$ is compatible [14] with the natural order relation in Λ.

LEMMA 1. *If A is a normal subset of the Köthe space Λ, then A^0 is a normal subset of Λ^*.*

PROOF. Observe first that

$$\sup_{f \in A} \left| \int fg d\mu \right| = \sup_{f \in A} \int |fg| \, d\mu$$

for any g in A^0, that is, if g is in A^0, then $|g|$ is in A^0. If h is in Ω and $|h| \leqq |g|$ for some g in A^0, then

$$1 \geqq \sup_{f \in A} \int |fg| \, d\mu \geqq \sup_{f \in A} \int |fh| \, d\mu.$$

Hence h is in A^0. It follows that A^0 is normal.

THEOREM 2. *Suppose that A is a weakly bounded subset of the Köthe space Λ. If $\tilde{A}$ is the normal convex closure of A, then $(\tilde{A})^{00}$ satisfies condition* (*).

PROOF. The set $(\tilde{A})^{00}$ is by [4] weakly bounded and by Lemma 1 normal. From previous remarks $(\tilde{A})^{00}$ is also convex and weakly closed.

Suppose that (f_n) is any nondecreasing sequence of nonnegative functions in $(\tilde{A})^{00}$. For any function g in Λ^* there exists a positive constant $M(g)$ such that

$$\int f_n |g| \, d\mu \leqq M(g), \qquad n = 1, 2, \cdots.$$

Let $f = \lim f_n$ so that $f|g| = \lim f_n |g|$. By Fatou's lemma

$$\int f|g| \, d\mu \leqq \liminf \int f_n |g| \, d\mu \leqq M(g). \tag{1}$$

Since the characteristic functions of compact subsets of E are in the associated space Λ^*, the function f is locally integrable, and hence is in $\Omega(E, \mu)$. As the function g was chosen arbitrarily from Λ^*, (1) shows that f is in Λ. It remains to show that f is a weak limit of the sequence (f_n), and hence is in $(\tilde{A})^{00}$.

To this end, suppose that a positive number ϵ and a function g from Λ^* have been chosen arbitrarily. Because $f|g|$ is integrable, there exists a compact subset K of E, such that

$$\int_{C(K)} f|g| \, d\mu \leqq \frac{\epsilon}{6} \cdot$$

A positive number δ exists such that for every measurable subset F of K for which $\mu(F) < \delta$,

$$\int_F f|g| \, d\mu < \frac{\epsilon}{6} \cdot$$

By Egoroff's theorem, a measurable subset F of K exists, such that $\mu(F) < \delta$, and f_n converges uniformly to f on $K \sim F$. Choose a number N such that $n \geqq N$ implies that $|fg - f_n g| < \epsilon/3\mu(K \sim F)$. It follows that

$$\left|\int (f - f_n)gd\mu\right| \leqq \int |fg - f_ng| \, d\mu < \epsilon, \qquad n \geqq N.$$

As ϵ and g were chosen arbitrarily,

$$\lim_{n\to\infty} \left|\int (f - f_n)gd\mu\right| = 0$$

for every g in Λ^*. Hence f is the weak limit of the sequence (f_n).

COROLLARY 1. *A normal subset A of the Köthe space Λ is weakly bounded if and only if $(\tilde{A})^{00}$ satisfies condition* (*).

One obtains from this corollary that $\mathcal{K}_w(\Lambda, \Lambda^*) = S(\Lambda, \Lambda^*)$, if W contains all admissible subsets of Λ^*.

4. It is known: if X and Y are two linear spaces in duality, then every weakly bounded subset of X is bounded for the Mackey topology $m(X, Y)$, [**2**]. For a Köthe space Λ it was shown by Dieudonné in [**4**] that the bounded subsets for the weak topology $\sigma(\Lambda, \Lambda^*)$ are the same as the bounded subsets for the strong topology. His proof is based on results of Mackey [**12**; **13**]. A simple direct proof will now be given.

LEMMA 2. *Every weakly bounded subset of the Köthe space Λ is bounded for the strong topology $S(\Lambda, \Lambda^*)$.*

PROOF. Suppose the weakly bounded subset A of Λ is not bounded for the topology $S(\Lambda, \Lambda^*)$. By Corollary 1 an admissible subset B of Λ^* and a sequence (f_n) in A exist such that for each integer n a function g_n in B can be found for which

$$\int |g_nf_n| \, d\mu \geqq n2^n.$$

Set $h_n = \sum_{k=1}^{n} 2^{-k}|g_k|$ for $n = 1, 2, \cdots$. As (h_n) is a nondecreasing sequence of nonnegative functions in B, there exists a function h in B such that $h \geqq h_n$ for $n = 1, 2, \cdots$. The normal closure $\overline{A}$ of A is weakly bounded [**4**], but

$$\sup_{f\in\overline{A}} \left|\int fhd\mu\right| \geqq \int |f_nh| \, d\mu \geqq n$$

for $n = 1, 2, \cdots$ implies that $\overline{A}$ is not weakly bounded. Hence the assumption that A is not bounded for the topology $S(\Lambda, \Lambda^*)$ is false.

Lemma 3. *If A is an absorbing admissible subset of the Köthe space Λ, then A^{00} is a bounded neighborhood of zero for the strong topology.*

Proof. Because A is normal and absorbing A^0 is normal and weakly bounded. It follows that A^{00} is a neighborhood of zero for the topology $S(\Lambda, \Lambda^*)$.

Since A is admissible, it is by Theorem 1 weakly bounded. Therefore, A^0 is absorbing. But A^0 being absorbing implies that A^{00} is weakly bounded. It follows from Lemma 2 that A^{00} is bounded for the strong topology.

Theorem 3. *The Köthe space Λ is a Banach space for the strong topology if and only if Λ contains an absorbing admissible set.*

Proof. Suppose that A is an absorbing admissible subset of Λ, then A^{00} is a convex bounded neighborhood of zero for the strong topology $S(\Lambda, \Lambda^*)$. By a theorem of Kolmogoroff [9] this topology is equivalent to a norm topology. Since Λ is complete for $S(\Lambda, \Lambda^*)$ [5], it is a Banach space.

Conversely if Λ is a Banach space for $S(\Lambda, \Lambda^*)$ then the unit sphere A in Λ is a weakly bounded absorbing set. It follows from Theorem 2 that $(A)^{00}$ is an absorbing admissible subset of Λ.

Corollary 2. *The Köthe space Λ is a Banach space for the topology $S(\Lambda, \Lambda^*)$ if and only if Λ^* is a Banach space for the topology $S(\Lambda^*, \Lambda)$.*

Proof. Suppose that Λ is a Banach space for the topology $S(\Lambda, \Lambda^*)$ and let A be an absorbing admissible subset of Λ. By Lemma 1, A^0 is normal. As A is weakly bounded and absorbing, A^0 is absorbing and weakly bounded. It is clear that A^0 is convex; therefore, A^0 is an admissible absorbing subset of Λ^*. Hence Λ^* is a Banach space.

The converse is proved by interchanging the roles of Λ and Λ^* in the preceding argument.

Corollary 2 shows that the statement of Theorem 3 could be made in terms of the subsets of Λ^*. With this alternate approach in mind one states:

Theorem 4. *The Köthe space Λ is a Fréchet space for the strong topology if and only if Λ^* contains a sequence $B_1 \subset B_2 \subset \cdots$ of admissible subsets, such that every weakly bounded subset B of Λ^* is contained in one of the B_k.*

Proof. Suppose Λ^* contains such a sequence $B_1 \subset B_2 \subset \cdots$ of admissible sets. The topology of uniform convergence on these sets is clearly the strong topology for Λ. As this topology has a countable base for the neighborhood system of zero, it is a metric topology. The completeness of Λ with this topology follows from [**5**].

Conversely, suppose that Λ is a Fréchet space for the strong topology. A sequence $A_1 \supset A_2 \supset \cdots$ of normal convex subsets of Λ can be found, which is a base in this topology for the neighborhood system of zero. The sets in the sequence $A_1^0 \subset A_2^0 \subset \cdots$ are admissible, for, each A_i is both normal and absorbing. This sequence satisfies the further condition that each weakly bounded subset B of Λ^* is contained in one of the A_k^0. For, if B is a weakly bounded subset of Λ^*, and $\tilde{B}$ is its normal closure, then $\tilde{B}^0$ is a neighborhood of zero in Λ. This implies an integer k can be found such that $A_k \subset \tilde{B}^0$. Hence $B \subset A_k^0$.

REMARK. If Λ is a Fréchet space with the strong topology, Corollary 2 suggests asking, is Λ^* a Fréchet space with its strong topology. The following discussion shows that if Λ is a Fréchet space, but not a Banach space, then $S(\Lambda^*, \Lambda)$ is not a metric topology. Suppose Λ^* with $S(\Lambda^*, \Lambda)$ is a Fréchet space even though Λ with $S(\Lambda, \Lambda^*)$ is not a Banach space. By Theorem 4 a sequence $B_1 \subset B_2 \subset \cdots$ of admissible subsets of Λ^* can be found whose union is Λ^*. The sets B_1^{00}, $B_2^{00}, \cdots$ are again admissible, and in addition are strongly closed. As the union of the sets B_i^{00} is again Λ^*, the Baire category theorem ensures the existence of an integer k such that B_k^{00} has an interior point. Thus B_k^{00} is an absorbing admissible set, that is, Λ^* is a Banach space. By Corollary 2 this implies that Λ is a Banach space, contrary to assumption. Thus, $S(\Lambda^*, \Lambda)$ is not a metric topology.

5. A strong unit u in a partially ordered vector space X is an element such that for each x in X, there exists a constant λ for which $x \leqq \lambda u$.

THEOREM 5. *If the Köthe space Λ^* contains a strong unit, then Λ has a unique Köthe topology, and with this topology Λ is a Banach space.*

PROOF. Let $u \geqq 0$ be a strong unit in Λ^*. Set $B = \{g \in \Lambda^* : |g| \leqq u\}$. If g is in Λ^*, then a constant λ exists such that g is in λB, that is, B is an absorbing subset of Λ^*. Clearly the set B is admissible. Hence Λ is a Banach space for the topology $S(\Lambda, \Lambda^*)$.

Let $\mathcal{K}_w(\Lambda, \Lambda^*)$ be any Köthe topology for Λ. There exists a normal set V in W such that u is in V. Thus $B \subset V$, from which it follows that $\mathcal{K}_w(\Lambda, \Lambda^*)$ is finer than $S(\Lambda, \Lambda^*)$. As $S(\Lambda, \Lambda^*)$ is always finer than $\mathcal{K}_w(\Lambda, \Lambda^*)$, these two topologies must be the same.

The existence of a unique Köthe topology on Λ does not imply that Λ^* has a strong unit. For consider the Köthe space Ω of all real sequences. Its associate, Φ, is the space of sequences which are zero except at a finite number of places. This space has no strong unit. However, Ω has only one Köthe topology. Note that the weak topology for Ω, that is the topology of pointwise convergence, is also the weakest Köthe topology. This topology is compatible with the order relation in Ω and is also a metric topology. Hence by [6] or [8] it is the finest compatible topology. Since $S(\Omega, \Phi)$ is a compatible topology it is weaker than the weak topology $\sigma(\Omega, \Phi)$. Thus these two topologies are the same, that is, Ω has only one Köthe topology.

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