VARIATIONAL COMPLETENESS FOR COMPACT SYMMETRIC SPACES

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We prove the following theorem: *Let K be a connected symmetric subgroup of the group of isometries of a compact connected globally Riemannian symmetric space M. Then, K acts in a variationally complete manner on M.* (We refer to the work of R. Bott and H. Samelson [1; 2] for a definition of this concept and applications to topology.)

Let G be a compact, connected Lie group, L a connected symmetric subgroup of G. Bott and Samelson have proved the following: (1) L acts in a variationally complete manner on G/L, the right coset space, (2) L acts via the linear isotropy representation in a variationally complete way on the tangent space at a point of G/L.

Our theorem then generalizes (1), but does not imply (2). The method is essentially the same as Bott's [1], but uses a Lie algebra approach in a stronger way. The proof then provides an exposition of Bott's very important result from a Lie-algebra point of view. Notice also that it suggests the general program of studying the Jacobi-fields on a symmetric space as a problem of Lie algebra theory, even in the nonvariational complete case.

1. We briefly recall Bott's notations: If $p\in M$, M_p denotes the tangent space to M at p. Let $O_p(K)$ be the orbit of K at p, let $O(K)_p$ be its tangent space at p, and let $g:R\to M$ be a geodesic of M beginning at p, perpendicular to $O_p(K)$. Consider the vector space J_g of Jacobi vector fields along g. A Jacobi field is a map $t\to Y_t\in M_g(t)$ for $t\in R$ that is a solution of the Jacobi Equation:

$$Y_t'' + R(X_t, Y_t)(X_t) = 0, \tag{1.1}$$

where $t\to X_t$ is the tangent vector field to the geodesic g, $R(X_t, Y_t)$ is the linear map $M_{g(t)}\to M_{g(t)}$ defined by the curvature form evaluated at (X_t, Y_t) and $t\to Y_t''$ is the second covariant derivative of the field Y_t along g.

J_g^K denotes the focal subspace of J_g relative to $O(K)$, i.e. J_g^K consists of those fields Y_t satisfying the initial conditions

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$$(1.2)\qquad \begin{aligned} &Y_0 \in O(K)_p, \\ &Y_0' + T_g(Y_0) \in O(K)_p^{\perp}, \end{aligned} \quad \text{where } T_g \text{ is a certain linear}$$

transformation $O(K)_p \to O(K)_p$.

Hence, $\dim J_g^K = \dim M$.

Let $\boldsymbol{K}$ denote the Lie algebra of K. There is a linear mapping $\pi: \boldsymbol{K} \to J_g^K$ such that for $k \in \boldsymbol{K}$, $\pi(k)$ is obtained by restricting the vector field determined by k on M to g.

By definition, to prove variational completeness we must show that every element of J_g^K that vanishes at some point of g must lie in $\pi(\boldsymbol{K})$.

2. M is a coset space G/L, where L is a compact, symmetric subgroup of a compact, connected Lie group G. It evidently suffices to prove the theorem in the case where p is the identity coset.

We have a natural reduction $\boldsymbol{G} = \boldsymbol{L} \oplus \boldsymbol{M}$, with $[\boldsymbol{M}, \boldsymbol{M}] \subset \boldsymbol{L}$ and $[\boldsymbol{L}, \boldsymbol{M}] \subset \boldsymbol{M}$. (For the ideas and results of the differential geometry of symmetric spaces, see [**3**].) M_p can be identified with $\boldsymbol{M}$, and each $M_{g(t)}$, for $t \in R$, can be identified with M_p by parallel transport along g. There is then a correspondence between vector fields along g and curves $t \to Y_t$ in $\boldsymbol{M}$. The metric on M can be considered as induced by a positive definite quadratic form on $\boldsymbol{G}$ invariant under Ad G.

Let P denote the projection of $\boldsymbol{G}$ on $\boldsymbol{M}$. Then $\boldsymbol{N} = P(\boldsymbol{K})$ is identified with $O(K)_p$. Let $X \in \boldsymbol{M}$ correspond to X_0. Then $X \in \boldsymbol{N}^{\perp}$. One sees that a curve $t \to Y_t$ in $\boldsymbol{M}$ corresponds to a vector field in J_g^K if and only if

$$(2.1)\qquad \begin{aligned} &Y_0 \in \boldsymbol{N}, \\ &Y_0' + T_g(Y_0) \in \boldsymbol{N}^{\perp}, \end{aligned}$$

$$(2.2)\qquad Y_t'' = (\mathrm{Ad}\, X)^2 Y_t,$$

where $t \to Y_t'$ is now the derivative in the ordinary sense. (One uses the explicit formula $\mathrm{Ad}[x, y] = -R(x, y)$ for the curvature in symmetric spaces and the fact that curvature is invariant under parallel translation in identifying $R(X_t, Y_t)X_t$ with $(\mathrm{Ad}\, X)^2 Y_t$, [**3**].)

If $k \in \boldsymbol{K}$, $\pi(k)$ corresponds to the curve $t \to \pi(k)_t = P((\mathrm{Ad}\, \mathrm{Exp}\, tX)(k))$, hence $\pi(k)_0 = P(k)$, $\pi(k)_0' = P([X, k])$.

The question of variational completeness can now be treated in this Lie algebra setting as a property of solutions of vector-valued ordinary linear differential equations (2.2) with constant coefficients. In particular, a reduction of $\boldsymbol{M}$ into subspaces invariant under

$(\text{Ad } X)^2$ leads to a decomposition of solutions of (2.2) into solutions taking values in the invariant subspaces.

3. The proof.

(a) kernel $\pi = \boldsymbol{K} \cap \boldsymbol{L} \cap$ kernel Ad X.

(b) dim $\pi(\boldsymbol{K})$ = dim $\boldsymbol{K}$ − dim $\boldsymbol{K} \cap \boldsymbol{L} \cap$ kernel Ad X = dim $P(\boldsymbol{K})$ + dim $\boldsymbol{K} \cap \boldsymbol{L}$ − dim $\boldsymbol{K} \cap \boldsymbol{L} \cap$ kernel Ad X = dim $P(\boldsymbol{K})$ +dim Ad $X(\boldsymbol{K} \cap \boldsymbol{L})$.

(c) $[\boldsymbol{K}^{\perp}, \boldsymbol{K}^{\perp}) \subset \boldsymbol{K}$, $[\boldsymbol{K}, \boldsymbol{K}^{\perp}] \subset \boldsymbol{K}^{\perp}$, since $\boldsymbol{K}$ is a symmetric subalgebra of $\boldsymbol{G}$. (The perpendicular operation $\perp$ is always with respect to the given metric on $\boldsymbol{G}$.) $\boldsymbol{L}^{\perp} = \boldsymbol{M}$. $(\text{Ad}X)^2(P(\boldsymbol{K})) \subset P(\boldsymbol{K})$, since $X \in K^{\perp} \cap L^{\perp}$.

(d) Ad $X(\boldsymbol{K} \cap \boldsymbol{L}) \subset \boldsymbol{K}^{\perp} \cap \boldsymbol{L}^{\perp}$ hence Ad $X(\boldsymbol{K} \cap \boldsymbol{L}) \subset P(\boldsymbol{K})^{\perp} \cap \boldsymbol{M}$. $(\text{Ad } X)^2(\text{Ad } X(\boldsymbol{K} \cap \boldsymbol{L})) \subset (\text{Ad } X)^2(\boldsymbol{K}^{\perp} \cap \boldsymbol{L}^{\perp}) \subset \text{Ad } X(\boldsymbol{K} \cap \boldsymbol{L})$.

(e) Define $\boldsymbol{Q} = \boldsymbol{M} \cap P(\boldsymbol{K})^{\perp} \cap \text{Ad}\, X(\boldsymbol{K} \cap \boldsymbol{L})^{\perp} = (P(\boldsymbol{K}) + \text{Ad}\, X(\boldsymbol{K} \cap \boldsymbol{L}))^{\perp} \cap \boldsymbol{M}$. Then, $(\text{Ad } X)^2(\boldsymbol{Q}) \subset \boldsymbol{Q}$, $\boldsymbol{Q} \subset \boldsymbol{K}^{\perp} \cap \boldsymbol{L}^{\perp}$.

(f) $\text{Ad}^2\, X(\boldsymbol{Q}) = 0$, for, Ad $X(\boldsymbol{Q}) \subset \boldsymbol{K} \cap \boldsymbol{L}$, hence $\text{Ad}^2\, X(\boldsymbol{Q}) \subset \text{Ad } X(\boldsymbol{K} \cap \boldsymbol{L}) \cap \boldsymbol{Q} = 0$.

(g) Dim M = dim $\boldsymbol{M}$ = dim J_g^K = dim $(\pi(\boldsymbol{K}) + \boldsymbol{Q})$.

(h) If $k \in \boldsymbol{K}$, then $\pi(k)_t \in \boldsymbol{Q}^{\perp}$ for all $t \geqq 0$.

Proof. $\pi(k)_t = P(\sinh(\text{Ad } tX)(k)) + P(\cosh(\text{Ad } tX)(k))$. Now, $P(\cosh(\text{Ad } tX)(k)) \in P(\boldsymbol{K}) \subset \boldsymbol{Q}^{\perp}$. Then, $\sinh(\text{Ad } tX)(k) \in \boldsymbol{Q}^{\perp}$ because of (e) and (f), and the fact that $(\text{Ad } X)^2$ is a symmetric transformation, with respect to the positive-definite quadratic form on $\boldsymbol{G}$, that commutes with P.

Now, define a map $\psi: \boldsymbol{Q} \to J_g^K$ as follows: For $q \in \boldsymbol{Q}$, $\psi(q)_t$ is the curve in $\boldsymbol{M}$ satisfying (2.2) and

$$\psi(q_0) = 0,$$
$$\psi(q_0') = q.$$

Because of (e) and (f), $\psi(q)_t = tq$, and hence ψ is one-to-one. Then $J_g^K = \psi(\boldsymbol{Q}) + \pi(\boldsymbol{K})$, $\psi(\boldsymbol{Q}) \cap \pi(\boldsymbol{K}) = 0$ and the theorem follows. For if $t \to Y_t$ is an element of J_g^K, $Y_t = \pi(k)_t + tq$ for $k \subset \boldsymbol{K}$, $q \in \boldsymbol{Q}$, and $Y_t = 0$ for some $t > 0$, then $q = 0$ by (h).

Bibliography

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