THE UNIFORMIZATION OF A CLASS OF SIMPLY CONNECTED RIEMANN SURFACES

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After defining a class of simply connected Riemann surfaces, this study determines the type of a surface from this class and obtains the forms of the uniformizing function and its derivative.

Definition of class of surfaces. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two infinite sequences, and for each $n \geqq 1$ let $\{a_i(n)\}_{i=1}^{q_n}$ and $\{b_i(n)\}_{i=1}^{q_n}$ be finite sequences such that for $n \geqq 1$ and $1 \leqq i \leqq q_n - 1$:

$$0 < a_i(1) < b_i(1) < a_{i+1}(1) < b_{i+1}(1) < a_1;$$

for n even and $n \geqq 2$,

$$a_{n-1} < b_{n-1} < b_i(n) < a_i(n) < b_{i+1}(n) < a_{i+1}(n) < b_n < a_n;$$

for n odd and $n \geqq 3$,

$$b_{n-1} < a_{n-1} < a_i(n) < b_i(n) < a_{i+1}(n) < b_{i+1}(n) < a_n < b_n.$$

A surface F of the class to be discussed consists of sheets S_n and $S_i(n)$, $n=1, 2, 3, \cdots$, $i=1, 2, \cdots, q_n$, over the w-sphere where, for S_n and $S_i(n)$ copies of the w-sphere:

(a) S_1 is slit along the real axis from $a_i(1)$ to $b_i(1)$, $i=1, 2, \cdots, q_1$, and from a_1 to b_1. (b) for $n \geqq 1$, S_{2n} is slit along the real axis from a_{2n-1} to b_{2n-1}, from b_{2n} to a_{2n}, and from $b_i(2n)$ to $a_i(2n)$, $i=1, 2, \cdots, q_{2n}$. (c) for $n \geqq 1$, S_{2n+1} is slit along the real axis from b_{2n} to a_{2n}, from a_{2n+1} to b_{2n+1}, and from $a_i(2n+1)$ to $b_i(2n+1)$, $i=1, 2, \cdots, q_{2n+1}$. (d) for $n \geqq 1$ and $i=1, 2, \cdots, q_n$, $S_i(n)$ is slit along the real axis between $a_i(n)$ and $b_i(n)$. (e) S_n and S_{n+1} are joined along the slits between a_n and b_n to form first order branch points at the end points. (f) $S_i(n)$ is joined to S_n along the slits between $a_i(n)$ and $b_i(n)$ to form first order branch points at the end points.

A special case of this class was considered in [2]. However, only the type of a surface was established.

The uniformizing function. F is simply connected, and by the Fundamental Mapping Theorem there exists a unique function g which maps F in a one-to-one way onto $|z| < R \leqq \infty$, and where for $f(z) = g^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$. By slitting each sheet of F along the uncut part of the real axis, two surfaces of hyperbolic type are obtained, and an application of the reflection principle to the

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uniformizing function of one of these surfaces in the upper half plane shows that $f(z)$ is real for z real. Furthermore, the image of F in the z-plane satisfies the following properties. The image of S_n or $S_i(n)$ is a region symmetric about the real axis. S_1 and $S_i(1)$, $i=1, 2, \cdots, q_1$, are mapped onto a domain containing $z=0$ and bounded by a simple closed curve C_1 which intersects the real axis at α_1 and $-\beta_1$. For $n>1$, S_n and $S_i(n)$, $i=1, 2, \cdots, q_n$, are mapped onto an annular region about $z=0$ bounded by two simple closed curves C_{n-1} and C_n which intersect the real axis at $-\beta_{n-1}$ and α_{n-1}, and $-\beta_n$ and α_n, respectively. Each $S_i(n)$ is mapped onto a domain bounded by a simple closed curve $C_i(n)$ which intersects the real axis at $(-1)^{n+1}\beta_i(n)$ and $(-1)^{n+1}\alpha_i(n)$. If $(-1)^n\gamma_n$ and $(-1)^{n+1}\gamma_i(n)$ denote the image in the z-plane of $w=\infty$ for S_n and $S_i(n)$, while $(-1)^n\delta_n$ and $(-1)^{n+1}\delta_i(n)$ denote the image of $w=0$ for S_n and $S_i(n)$, respectively, then the following inequalities are satisfied:
for n even, $n\geqq 2$,

$$0 < \alpha_{n-1} < \delta_n < \gamma_n < \alpha_n < \alpha_{n+1};$$

for n odd, $n\geqq 3$,

$$0 > -\gamma_1 > -\beta_1 > -\beta_{n-1} > -\delta_n > -\gamma_n > -\beta_n > -\beta_{n+1};$$

for $1\leqq i\leqq q_1-1$,

$$0 < \alpha_i(1) < \delta_i(1) < \gamma_i(1) < \beta_i(1) < \alpha_{i+1}(1) < \beta_{i+1}(1) < \alpha_1;$$

for n even, $n\geqq 2$, $1\leqq i\leqq q_n-1$,

$$-\beta_{n-1} > -\beta_i(n) > -\delta_i(n) > -\gamma_i(n) > -\alpha_i(n) > -\beta_{i+1}(n) > -\alpha_{i+1}(n) > -\beta_n;$$

for n odd, $n\geqq 3$, $1\leqq i\leqq q_n-1$,

$$\alpha_{n-1} < \alpha_i(n) < \delta_i(n) < \gamma_i(n) < \beta_i(n) < \alpha_{i+1}(n) < \beta_{i+1}(n) < \alpha_n.$$

The approximating closed surfaces. Let F_n be the part of F formed from S_k, $k=1, 2, \cdots, 2n+1$, and $S_i(k)$, $k=1, 2, \cdots, 2n$ and $i=1, 2, \cdots, q_{2k}$, with the slits in S_{2n+1} from a_{2n+1} to b_{2n+1} and from $a_i(n+1)$ to $b_i(2n+1)$, $i=1, 2, \cdots, q_{2n+1}$, deleted.

NOTATION.

$$\overset{*}{\alpha}_\phi = 1 - z/\alpha_\phi. \qquad \overset{**}{\alpha}_\phi(k) = 1 + (-1)^k z/\alpha_\phi(k).$$
$$\overset{*}{\beta}_\phi = 1 + z/\beta_\phi. \qquad \overset{**}{\beta}_\phi(\mathrm{k}) = 1 + (-1)^k z/\beta_\phi(k).$$
$$\overset{*}{\gamma}_\phi = 1 - (-1)^k z/\gamma_\phi. \qquad \overset{**}{\gamma}_\phi(k) = 1 + (-1)^k z/\gamma_\phi(k).$$
$$\overset{*}{\delta}_\phi = 1 - (-1)^k z/\delta_\phi. \qquad \overset{**}{\delta}_\phi(k) = 1 + (-1)^k z/\delta_\phi(k).$$

LEMMA 1.

$$R_n(z) = [z/\overset{*}{\gamma}_{1,n}] \prod_{k=1}^{2n} [\overset{*}{\delta}_{k,n}/\overset{*}{\gamma}_{k,n}] \prod_{k=1}^{2n} \left[\prod_{i=1}^{q_k} \overset{**}{\delta}_{i,n}(k)/\overset{**}{\gamma}_{i,n}(k) \right] \overset{*}{\delta}_{2n+1,n},$$

where

$$R_n'(z) = \prod_{k=1}^{2n} \left[(\overset{*}{\alpha}_{k,n}\overset{*}{\beta}_{k,n}/\overset{*2}{\gamma}_{k,n}) \prod_{i=1}^{q_k} \overset{**}{\alpha}_{i,n}(k)\overset{**}{\beta}_{i,n}(k)/\overset{**}{\gamma}_{i,n}(k)^2 \right],$$

defines a mapping of the z-plane onto F_n.

PROOF. F_n is a simply connected closed surface and hence is the Riemann surface of the inverse of a unique rational function R_n such that $w=R_n(z)$, $R_n(0)=0\in S_1$, $R_n'(0)=1$, and $R_n(\infty)=\infty\in S_{2n+1}$.

LEMMA 2. *F is parabolic.*

PROOF. Suppose that F is not parabolic, and thus g is a mapping of F onto $\{|z|<R<\infty\}$. Let D_n be the z-plane slit along the real axis from $\alpha_{2n,2n}$ to $+\infty$. $\zeta=\psi_n(z)=g[R_n(z)]$ defines a schlicht mapping of D_n onto a simply connected region Δ_n of the ζ-plane bounded by C_{2n+1}, $C_i(2n+1)$, $i=1, 2, \cdots, q_{2n+1}$, and the segments $(\alpha_{2n}, \alpha_{2n+1})$ and $(-\beta_{2n+1}, -\gamma_{2n})$. Then $\alpha_{2n,2n}\leqq d(0, C_{2n+1})\leqq R<\infty$ where $d(0, C_{2n+1})$ is the distance from $\zeta=0$ to the curve C_{2n+1}, and thus there exists a subsequence $\{\alpha_{2n_i,2n_i}\}$ such that $\alpha_{2n_i,2n_i}\to A\leqq R$. Then ψ_{n_i} is a schlicht mapping of D_{n_i} into Δ_{n_i}, where for n_i sufficiently large, $\Delta_{n_i}\subset\{\zeta\,|\,|\zeta|\leqq R\}$. If D is the z-plane slit along the real axis from $z=A$ to $+\infty$, then $\{\psi_{n_i}\}$ forms a normal family by Montel's Theorem and $\psi_{n_j}(z)\to\psi(z)$ uniformly on any compact subset of D. ψ maps D schlichtly onto $\{\zeta\,|\,|\zeta|\leqq R\}$. Then $R_{n_j}(z)\to H(z)=f[\psi(z)]$, where H is a function meromorphic in D. $H(z)\not\equiv\infty$ because $R_n(0)=0$.

Let D^* be the z-plane slit along the real axis from $z=A$ to $-\infty$. Now $R_{n_j}(z)$ is defined in D^* and for j sufficiently large assumes no negative real values in any compact subset of D^*. Then $\{R_{n_j}\}$ is a normal family in D^*, so that a subsequence $\{R_m\}$ of $\{R_{n_j}\}$ converges uniformly on any compact subset of D^* to a function G. Because of a common domain of convergence, G is the analytic continuation of H. Then $w=G(z)$ defines a mapping of the z-plane punched at $z=A$ and ∞ one to one and conformally onto an open doubly connected Riemann surface F^* of which F is a subsurface obtained by inserting some slits in F^* over the real axis. Clearly this is impossible, and thus $R=\infty$.

LEMMA 3. $R_n(z)\to f(z)$ *uniformly on any compact subset of* $|z|<\infty$ *as* $n\to\infty$.

PROOF. Because F is parabolic and thus $\{\Delta_n\}$ converges in the sense of Carathéodory to its kernel $|\zeta|<\infty$, it follows that $z=R_n^{-1}[f(\zeta)]\to\zeta=g[R_n(z)]$ uniformly on any compact subset of $|\zeta|<\infty$. Then $\{D_n\}$ converges to $|z|<\infty$ and thus $R_n(z)\to f(z)$ uniformly on any compact subsets of $|z|<\infty$.

LEMMA 4. *$\alpha_{k,n}\to\alpha_k$, $\beta_{k,n}\to\beta_k$, $\gamma_{k,n}\to\gamma_k$, $\delta_{k,n}\to\delta_k$, $\alpha_{i,n}(k)\to\alpha_i(k)$, $\beta_{i,n}(k)\to\beta_i(k)$, $\gamma_{i,n}(k)\to\gamma_i(k)$, and $\delta_{i,n}(k)\to\delta_i(k)$ as $n\to\infty$.*

PROOF. This is a consequence of Hurwitz's Theorem.

LEMMA 5. *The infinite product*

$$\pi(z) = [z/\overset{*}{\gamma}_1]\left[\prod_{k=2}^{\infty}\overset{*}{\delta}_k/\overset{*}{\gamma}_k\right]\prod_{k=1}^{\infty}\left[\prod_{i=1}^{q_k}\overset{**}{\delta}_i(k)/\overset{**}{\gamma}_i(k)\right]$$

converges uniformly on any compact subset of $|z|<\infty$.

PROOF. $\pi(z)$ can be decomposed onto four infinite products similar to $M(z)=\prod_{k=1}^{\infty}\overset{*}{\delta}_{2k}/\overset{*}{\gamma}_{2k}$. Since F is parabolic and $\{\Delta_n\}\to\{|\zeta|<\infty\}$, then $\delta_k\to\infty$ and $\gamma_k\to\infty$ as $k\to\infty$. Therefore, for any $R>0$ there exists $n_0=n_0(R)$ such that $\delta_k>R$ and $\gamma_k>R$ for $k\geqq n_0$. Then consider $M_p(z)=\prod_{k=n_0}^{n_0+p}\overset{*}{\delta}_{2k}/\overset{*}{\gamma}_{2k}$. M_p is holomorphic and $M_p(z)\neq 0$ for $|z|\leqq R$. A sufficient condition for the uniform convergence of M_p in a region as $p\to\infty$ is the uniform convergence of

$$\sum_{k=n_0}^{n_0+p}\log(\overset{*}{\delta}_{2k}/\overset{*}{\gamma}_{2k})$$

in this region as $p\to\infty$, where the logarithm is the principal value. By the Cauchy criterion, this last series converges uniformly in $\{|z|\leqq R\}$ provided for $\epsilon>0$ and for all z such that $|z|\leqq R$ there exists $N(\epsilon)>0$ such that for $n>N(\epsilon)$ and for all $p>0$,

$$\left|\sum_{k=n_0+n}^{n_0+n+p}\log(\overset{*}{\delta}_{2k}/\overset{*}{\gamma}_{2k})\right|<\epsilon.$$

Because $\cdots\delta_{2k}<\gamma_{2k}<\delta_{2k+2}<\gamma_{2k+2},\cdots$, then for $m\geqq 1$ and $p>0$,

$$0<\sum_{k=n_0+n}^{n_0+n+p}[(1/\delta_{2k})^m-(1/\gamma_{2k})^m]<(1/\delta_{2n_0+2n})^m,$$

and thus

$$\left|\sum_{k=n_0+n}^{n_0+n+p}\log(\overset{*}{\delta}_{2k}/\overset{*}{\gamma}_{2k})\right| = \left|\sum_{m=1}^{\infty}[z^m/m]\left\{\sum_{k=n_0+n}^{n_0+n+p}[(1/\delta_{2k})^m-(1/\gamma_{2k})^m]\right\}\right|$$

$$\leqq\sum_{m=1}^{\infty}(R^m/m)(1/\delta_{2n_0+2n})^m\leqq\sum_{m=1}^{\infty}(R/\delta_{2n_0+2n})^m=R/(\delta_{2n_0+2n}-R).$$

Because $\delta_k \to \infty$ as $k \to \infty$, the required inequality is established. Hence $M(z)$ converges uniformly on any compact subset of the z-plane. Similarly, the other products into which $\pi(z)$ can be decomposed converge uniformly on any compact subset of the z-plane.

LEMMA 6. $\pi(z) = f(z)$.

PROOF. Because of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$. Because for $m \geqq 1$,

$$0 < \sum_{k=i}^{\infty} [(1/\delta_{2k})^m - (1/\gamma_{2k})^m] \leqq (1/\delta_{2i})^m$$

and

$$0 < \sum_{k=i}^{\infty} [(1/\delta_{2k,n})^m - (1/\gamma_{2k,n})^m] < (1/\delta_{2i,n})^m,$$

it follows with the aid of Lemma 4 that for $n_0 > 1$,

$$\begin{aligned} 0 &\leqq \limsup_{n\to\infty} \left| \sum_{k=1}^{n} [(1/\delta_{2k,n})^m - (1/\gamma_{2k,n})^m] - \sum_{k=1}^{\infty} [(1/\delta_{2k})^m - (1/\gamma_{2k})^m] \right| \\ &\leqq \limsup_{n\to\infty} \left| \sum_{k=n_0}^{n} [(1/\delta_{2k,n})^m - (1/\gamma_{2k,n})^m] - \sum_{k=n_0}^{\infty} [(1/\delta_{2k})^m - (1/\gamma_{2k})^m \right| \\ &< \lim_{n\to\infty} \left| (1/\delta_{2n_0,n})^m + (1/\delta_{2n_0})^m \right| = 2/(\delta_{2n_0})^m. \end{aligned}$$

Since $\delta_{2n_0} \to \infty$ as $n_0 \to \infty$,

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} [(1/\delta_{2k,n})^m - (1/\gamma_{2k,n})^m] - \sum_{k=1}^{\infty} [(1/\delta_{2k})^m - (1/\gamma_{2k})^m] \right\} = 0.$$

Thus in this manner all the coefficients of the Taylor expansion of $\log [R_n(z)/\pi(z)]$ about $z = 0$ can be shown to have a limit of zero as $n \to \infty$, and thus $\lim_{n\to\infty} \log [R_n(z)/\pi(z)] = 0$ which implies that $\lim_{n\to\infty} R_n(z) = f(z) = \pi(z)$.

LEMMA 7. *The infinite product*

$$P(z) = \prod_{k=1}^{\infty} \{\overset{*}{\alpha}_k \overset{*}{\beta}_k/(\overset{*}{\gamma}_k)^2\} \left\{ \prod_{i=1}^{q_k} \overset{**}{\alpha}_i(k) \overset{**}{\beta}_i(k)/[\overset{**}{\gamma}_i(k)]^2 \right\}$$

converges uniformly on any compact subset of the z-plane.

PROOF. $P(z)$ can be decomposed into four infinite products similar to $Q(z) = \prod_{k=1}^{\infty} \overset{*}{\alpha}_{2k} \overset{*}{\alpha}_{2k+1}/(\overset{*}{\gamma}_{2k+2})^2$. Because for $m \geqq 1$ and $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} [(1/\alpha_{2k})^m + (1/\alpha_{2k+1})^m - 2/(\gamma_{2k+2})^m] < 2/(\alpha_{2n_0+2n})^m,$$

an application of the Cauchy criterion similar to that described in the proof of Lemma 5 establishes the convergence of $Q(z)$. Similarly, the other factors of $P(z)$ converge.

LEMMA 8. $P(z)=f'(z)$.

PROOF. Because for all $i>0$ and $m\geqq 1$,

$$0 < \sum_{k=i}^{n} (1/\alpha_{2k})^m + \sum_{k=i}^{n-1} (1/\alpha_{2k+1,n})^m - \sum_{k=i}^{n-1} 2/(\gamma_{2k+2,n})^m < 2/(\alpha_{2i,n})^m,$$

$$0 < \sum_{k=i}^{\infty} [(1/\alpha_{2k})^m + (1/\alpha_{2k+1})^m - 2/(\gamma_{2k+2})^m] < 2/(\alpha_{2i})^m,$$

and similar inequalities, it follows in a manner similar to the proof of Lemma 6 that $\lim_{n\to\infty} \log [R_n'(z)/P(z)]=0$ and thus $P(z)=f'(z)$.

These lemmas prove the following theorem.

A Riemann surface of the class defined is parabolic and its mapping function f is given by

$$f(z) = [z/\overset{*}{\gamma}_1] \prod_{k=2}^{\infty} [\overset{*}{\delta}_k/\overset{*}{\gamma}_k] \prod_{k=1}^{\infty} \left[\prod_{i=1}^{q_k} \overset{**}{\delta}_i(k)/\overset{**}{\gamma}_i(k) \right]$$

where

$$f'(z) = \prod_{k=1}^{\infty} \left\{ [\overset{*}{\alpha}_k \overset{*}{\beta}_k/(\overset{*}{\gamma}_k)^2] \left[\prod_{i=1}^{q_k} \overset{**}{\alpha}_i(k) \overset{**}{\beta}_i(k)/(\overset{**}{\gamma}_i(k))^2 \right] \right\}.$$

The α, β, γ, and δ are ordered as indicated at the beginning of the paper.

It should be noted that the factors in the infinite products in Lemmas 5 and 7 are quotients. Hence the representations in this theorem are products of quotients, not quotients of products.

REFERENCES

1. G. R. MacLane, *Riemann surfaces and asymptotic values associated with real entire functions*, The Rice Institute Pamphlet, 1952.

2. H. E. Taylor, *Determination of the type and properties of the mapping functions for certain classes of Riemann surfaces*, Ph.D. Thesis, The Rice Institute, 1950.

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