NEIGHBORHOOD EXTENSIONS OF CONTINUOUS SELECTIONS

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Hanner (see [2, Theorem 19.2]) has proved that the property of being a neighborhood extension space with respect to the class of paracompact spaces is a local property. Theorem 1 of this paper generalizes Hanner's result, and states roughly that if a continuous selection can be locally extended to a continuous selection, then it has a neighborhood extension which is also a continuous selection. The proof of Theorem 1 differs considerably from Hanner's proof, and seems at least as simple.

Let X be a paracompact space and Y a topological space. We assume that for each $x \in X$, $F(x)$ is a subset of Y. If K is a subset of X and ϕ is a mapping (continuous function) on K into Y, then ϕ is a selection for F if $\phi(x) \in F(x)$ for each $x \in K$. A subset S_1 of Y is an *F-neighborhood extension set* with respect to a subset S_2 of X if each mapping ϕ: $C \to S_1$, ϕ a selection for F, $C \subset S_2$ and C closed relative to S_2, can be extended to a mapping ψ: $G \to S_1$, where ψ is a selection for F, $G \subset S_2$ and G is open relative to S_2.

We assume that A is a closed subset of X and that f is a selection for F which maps A into Y.

THEOREM 1. *If for each $x \in A$ there exist neighborhoods V_x of x and U_x of $f(x)$ such that U_x is an F-neighborhood extension set with respect to $\overline{V}_x$, and $f[\overline{V}_x \cap A] \subset U_x$, then there exists an open set $N \supset A$ and a mapping g: $N \to Y$ such that $g|A = f$ and g is a selection for F.*

PROOF. Let $\mathcal{G}_1 = \{W | W = X - A$ or $W = V_x$ for some $x \in A\}$. Since X is paracompact, there is a σ-discrete refinement $\mathcal{G}_2$ of $\mathcal{G}_1$. We may assume without loss of generality that $\{\overline{W} | W \in \mathcal{G}_2\}$ is also a refinement of $\mathcal{G}_1$. We let $\mathcal{R} = \{W | W \in \mathcal{G}_2$ and $W \cap A \neq \varnothing\}$. It is possible to represent $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$, where each $\mathcal{R}_n$ is a discrete collection of open sets.

We wish to define inductively a sequence $A_0 \subset A_1 \subset A_2 \subset \cdots$ of closed sets and mappings g_k: $A_k \to Y$ such that:

(i) $A_0 = A$ and $g_0 = g$,

(ii) if $W \in \mathcal{R}_k$, then $W \cap A$ is contained in the interior of A_{k+1},

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(iii) each g_k is a selection for F and $g_{k+1}|A_k = g_k$ for $k=0, 1, 2, \cdots$.

We define A_0 and g_0 so as to satisfy (i). Now let us assume that $A_0 \subset A_1 \subset \cdots \subset A_n$ and $g_0, g_1, \cdots, g_n$ have been defined so as to satisfy the above conditions. We must define A_{n+1} and g_{n+1}.

Let W be any member of $\mathfrak{R}_{n+1}$. There exists $x \in A$ such that $\overline{W} \subset V_x$. The set $g_n^{-1}[U_x]$ is open relative to A_n, and thus there exists an open set W_1 such that $g_n^{-1}[U_x] = A_n \cap W_1$. Since $g_n[\overline{W} \cap A] = f[\overline{W} \cap A] \subset f[\overline{V}_x \cap A] \subset U_x$, it follows that $\overline{W} \cap A \subset W_1$. By normality of X, there exists an open set W_2 such that $\overline{W} \cap A \subset W_2$ and $\overline{W}_2 \subset W_1$. If we define $W_3 = W \cap W_2$, then $g_n[\overline{W}_3 \cap A_n] \subset U_x$ and we may extend $g_n | (\overline{W}_3 \cap A_n)$ to a mapping $\psi_W\colon W_4 \to U_x$, where W_4 is an open subset of $\overline{V}_x$ and ψ_W is a selection for F. There exists an open set W_5 such that $\overline{W}_3 \cap A_n \subset W_5$ and $\overline{W}_5 \subset W_4$.

We now define

$$A_{n+1} = A_n \cup \bigcup_{W \in \mathfrak{R}_n} (\overline{W}_5 \cap \overline{W}_3),$$

and

$$g_{n+1}(x) = \begin{cases} g_n(x), & \text{for } x \in A_n, \\ \psi_W(x), & \text{for } x \in \overline{W}_5 \cap \overline{W}_3 \text{ for some } W \in \mathfrak{R}_n. \end{cases}$$

The fact that $\mathfrak{R}_n$ is discrete implies that the set A_{n+1} is closed. It is easy to see that for each $W \in \mathfrak{R}_n$, $W \cap A \subset W_3 \cap W_5$ and hence $W \cap A$ is contained in the interior of A_{n+1}. The function g_{n+1} is well defined, and its continuity follows from the fact that $\mathfrak{R}_n$ is a discrete collection of open sets. It is also obvious that g_{n+1} is a selection for F.

We now obtain N and g by letting $N = \bigcup_{n=1}^{\infty}$ (interior A_n), and defining $g(x) = g_n(x)$ for $x \in$ (interior of A_n).

The preceding theorem is of interest in connection with the problem (Q_2) stated by Michael in [**4**]. The next theorem, which is a combination lifting and extension theorem, is of the same general nature as Steenrod's existence theorem on cross sections (see [**5**, p. 55]). A special case of this result was used by the author in [**1**].

We now let Y be a fiber space (in the sense of Hilton, see [**3**, p. 46]) with base space B, fiber K, and projection $p\colon Y \to B$. It is again assumed that X is a paracompact space and that A is a closed subset of X. We assume mappings $f\colon A \to Y$ and $h\colon X \to B$ such that $pf = h|A$. The definition of ANR (absolute neighborhood retract) is given in [**2**].

THEOREM 2. *If K is an ANR, then there exists an open set $N \supset A$ and a mapping $g\colon N \to Y$ such that $g|A = f$ and $pg = h|N$.*

PROOF. We define $F(x)=p^{-1}h(x)$ for each $x\in X$. For each $x\in A$, let W_x be a neighborhood of $h(x)$ such that there exists a homeomorphism η_x on $W_x\times K$ onto $p^{-1}[W_x]$ for which $p\eta_x(b, k)=b$. We define $U_x=Y$, and choose V_x to be any neighborhood of x for which $h[\overline{V}_x]\subset W_x$.

We will show that each U_x is an F-neighborhood extension set with respect to $\overline{V}_x$. Thus, we consider a closed subset C of $\overline{V}_x$ and a selection for F, $\phi: C\rightarrow U_x$. The mapping $\eta_x^{-1}\phi$ maps C into $W_x\times K$, and hence we can represent $\eta_x^{-1}\phi(t)=(\alpha(t), \beta(t))$, where α and β are mappings on C into W_x and K respectively. Since K is an ANR, and $\overline{V}_x$ is normal we can extend β to a mapping $\beta': G_1\rightarrow K$, where G_1 is open relative to $\overline{V}_x$. Since $\alpha(t)=p\eta_x(\alpha(t), \beta(t))=p\eta_x\eta_x^{-1}\phi(t)=p\phi(t)=h(t)$ for $t\in C$, we can let $G_2=\overline{V}_x\cap h^{-1}[W_x]$ and extend α to $\alpha': G_2\rightarrow W_x$ by letting $\alpha'=h|G_2$. We now obtain the desired selection for F by letting $G=G_1\cap G_2$ and defining

$$\psi(t) = \eta_x((\alpha'(t), \beta'(t))), \qquad \text{for } t\in G.$$

It is now easy to verify that the hypotheses of Theorem 1 are satisfied. Thus, there exists an open set $N\supset A$ and a mapping $g: N\rightarrow Y$ such that $g|A=f$ and g is a selection for F. The fact that g is a selection for F, however, means that $pg=h|N$, and this concludes the proof of Theorem 2.

A space Y has the neighborhood extension property with respect to the class of all paracompact spaces if corresponding to each paracompact space X and mapping f of a closed subset of X into Y there is an extension of f whose domain is an open set. We let M be the set of all spaces Y which have the neighborhood extension property with respect to the class of all paracompact spaces. The following theorem, which we prove by applying Theorem 1, seems to have been first demonstrated by Hanner (see [**2**, p. 340]).

THEOREM 3. *If each point in a space Y has a neighborhood which belongs to M, then Y belongs to M.*

PROOF. Let X be a paracompact space, let A be a closed subset of X, and let f be a mapping of A into Y.

We define $F(x)=Y$ for each $x\in A$. For each $x\in A$, we choose a neighborhood U_x of $f(x)$ such that $U_x\in M$. We may now choose a neighborhood V_x of x such that $f[\overline{V}_x\cap A]\subset U_x$. Since $\overline{V}_x$ is a paracompact space and $U_x\in M$, it follows that U_x is an F-neighborhood extension set with respect to $\overline{V}_x$. Thus, by Theorem 1 we can extend f to a mapping whose domain is an open set. This proves that $Y\in M$.

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A REMARK ON THE CHARACTERIZATION OF HOMOTHETIC TRANSFORMATION AND INVERSION[1]

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Given two closed orientable C^2 surfaces S, $\bar{S}$ in E^3, let $h: S \to \bar{S}$ be a differentiable homeomorphism, such that referring to a suitable origin O

$$\bar{X} = kX, \tag{1}$$

with k nonzero and C^2 at points at which $X \neq 0$, $\bar{X} \neq 0$. Write $p = -X \cdot N$, $\bar{p} = -\bar{X} \cdot \bar{N}$. Suppose further that S, $\bar{S}$ contains no pieces of cones with vertex O. We use the form $b = (\bar{N} \times X) \cdot dX$, introduced in [1]. We generalize results of [1] to

THEOREM 1′. *If* $\iint_S (H \mp k\bar{H}) p dA = 0$, *then* $k = const.$, *i.e. the map* h *is a homothetic transformation with center O.*

PROOF. The formulae $\bar{p} d\bar{A} = k^3 p dA$ and $db = (2/k^2) \bar{p} \bar{H} d\bar{A} + 2\bar{N} \cdot N dA$ yield $\iint_S N \cdot \bar{N} dA = -\iint_S k\bar{H} p dA$. Combining with $\iint_S dA = \iint_S H p dA$ gives

$$\iint_S (1 \mp N \cdot \bar{N}) dA = \iint_S (H \pm k\bar{H}) p dA,$$

and the theorem follows as in [1].

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