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CONDITIONS IMPLYING CONTINUITY OF FUNCTIONS

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In the study of functions on certain types of spaces, the question naturally arises as to what additional conditions may imply that the functions are continuous. Several papers, mainly [2; 3; 4], have considered this problem. In this note, some further results of this type are developed.

To avoid repetition, a function $f$ will be at least on a Hausdorff space $X$ onto a Hausdorff space $Y$ with additional restrictions stated as needed. Also $f$ is compact preserving (connected) if when $K$ is a compact (connected) subset of $X$, $f(K)$ is a compact (connected) subset of $Y$; $f$ has closed point inverses if for each $y \in Y$, $f^{-1}(y)$ is closed and $f$ is monotone if $f^{-1}(y)$ is connected. The rest of the terminology is standard.

In [1], it was shown that if $X$ is regular, $Y$ compact and if $f$ is closed with closed point inverses, $f$ is continuous. Combining this with Theorem 3.1 of [4], one has the result:

**Theorem 1.** If $f$ is a closed monotone connected function on a regular space $X$ onto a compact space $Y$, then $f$ is continuous.

It is easy to see that without the assumption that $Y$ is compact, the conclusion need no longer hold.

**Theorem 2.** If $X$ is locally compact, then if $f$ is compact preserving and point inverses are closed, $f$ is continuous.

Consider any point $x \in X$. Since $X$ is locally compact, $x$ has a neighborhood $U_0$ with a compact closure $\text{Cl} \ U_0$. Because continuity is a local property, one need only consider $f$ restricted to $\text{Cl} \ U_0$. On $\text{Cl} \ U_0$, $f$ is closed and $f(\text{Cl} \ U_0)$ is compact. Hence the conditions of Theorem 3 of [1] are satisfied and $f$ is continuous at $x$.

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If $X$ is not locally compact, then $f$ may not be continuous. The function in the first example on [3, p. 162] is such an instance.

**Definition 1.** A function $f$ has at worst a removable discontinuity at $x \in X$ if there is a $y \in Y$ such that for each neighborhood $V$ of $y$, there is a neighborhood $U$ of $x$ such that $f(U - \{x\}) \subseteq V$.

If $X$ satisfies the first axiom of countability, this definition is equivalent to Definition 3.2 of [4]. With this interpretation, the conditions of Theorem 3.6 of [4] may be relaxed somewhat.

**Theorem 3.** If $X$ is locally connected and $f$ is connected, then $f$ is continuous at $x_0$ if and only if $f$ has at worst a removable discontinuity at $x_0$.

With only minor change, the proof given by Pervin and Levine applies here.

**Theorem 4.** If $X$ is regular and $f$ is a closed function with closed point inverses, then if $f$ has a removable discontinuity at $x_0 \in X$, $f$ is continuous at $x_0$.

If $x_0$ is isolated in $X$, the result is obviously true. Assume that $x_0$ is nonisolated and $f$ is not continuous at $x_0$. Let $y$ be the point of $Y$ determined by the hypothesis. Since $y \not \in f(x_0)$ and $f^{-1}(y)$ is closed, a neighborhood $U$ of $x_0$ exists such that $f^{-1}(y) \cap \text{Cl } U = \emptyset$. Then $y \in f(\text{Cl } U)$ and because $f(\text{Cl } U)$ is closed, a neighborhood $V$ of $y$ exists for which $V \cap f(\text{Cl } U) = \emptyset$. There is a neighborhood $W$ of $x_0$ such that $f(W - \{x_0\}) \subseteq V$. Since $x_0$ is nonisolated, $U \cap W - \{x_0\} \neq \emptyset$. Hence $\emptyset \neq f(W - \{x_0\}) \cap f(\text{Cl } U) \subseteq V \cap f(\text{Cl } U)$, a contradiction.

**Definition 2.** A space $X$ will be said to have property $K$ at a point $x$ if for each infinite subset $A$ having $x$ as an accumulation point, there is a compact subset of $A \cup \{x\}$ which has $x$ as an accumulation point.

**Theorem 5.** If $X$ has property $K$ at $x_0$, then if $f$ is compact preserving and has closed point inverses, $f$ is continuous at $x_0$.

It can be assumed that $x_0$ is nonisolated. Suppose $f$ is not continuous at $x_0$ and that $\mathcal{U}$ is the family of neighborhoods of $x_0$. Then for some neighborhood $V$ of $f(x_0)$ and for each $U \in \mathcal{U}$, there is an $x_u$ such that $x_u \in U \cap f^{-1}(Y - V)$. Let $A = \{x_u \mid U \in \mathcal{U}\}$. Then $A$ is infinite, for $x_0$ is an accumulation point of $A$. By hypothesis, there is an infinite compact subset $K$ of $A \cup \{x_0\}$. By Theorem 2, $f$ restricted to $K$ is continuous. However $f(K - \{x_0\}) \subseteq Y - V$ but $f(x_0) \in V$, a contradiction.\(^1\)

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\(^1\) This is the referee's revision of the author's original proof.
Theorem 6. If $X$ is locally connected with property $K$ at each point and if $f$ is compact preserving and connected, then $f$ is continuous.

It need only be shown that point inverses are closed.

Let $y_0 \in Y$ and suppose $x_0 \in C f^{-1}(y_0) - f^{-1}(y_0)$. Denote the family of connected neighborhoods of $x_0$ by $C$ and the family of neighborhoods of $y_0$ by $U$. Select disjoint open neighborhoods $V$ and $U_0$ of $f(x_0)$ and $y_0$ respectively. For each $C \in C$ and $U \in U$, let the point $y(U, C) \in f(C) \cap U \cap U_0 - \{y_0\}$ and the point $x(U, C) \in f^{-1}(y(U, C)) \cap C$. The set $A$ of all such $x(U, C)$ is infinite and has $x_0$ as an accumulation point. By hypothesis, $A \cup \{x_0\}$ has an infinite compact subset $K$ with $x_0$ as an accumulation point. Note that $x_0 \in K$. Let $g$ denote the function $f$ restricted to $K$. Then $S = g(K) - \{g(x_0)\} = g(K) \cap (Y - V)$ is an infinite compact set and must have an accumulation point $z$. If $x = g^{-1}(z)$ is isolated in $K$, then $K - \{x\}$ and hence $S - \{z\}$ are compact, a contradiction. Assume then that for each accumulation point of $S$, its inverse in $K$ is an accumulation point of $K$.

Let $A$ be the set of accumulation points of $K$, excluding $x_0$. For each $x \in A$, select disjoint open neighborhoods $W_x$ and $R_x$ of $x$ and $x_0$ respectively. Each $K - W_x$ is compact and each $B_x = g(K - W_x) \cap S$ is a closed non-null subset of $S$. The family $\mathcal{B} = \{B_x | x \in A\}$ has the finite intersection property, for suppose the contrary. There would exist a finite number of neighborhoods $W_{x_1}, \ldots, W_{x_n}$ such that $K - \{x_0\} \subseteq \bigcup_{i=1}^n W_{x_i}$, but since for each $W_{x_i}$, there is a neighborhood $R_{x_i}$ of $x_0$ disjoint from $W_{x_i}$, $\bigcap_{i=1}^n R_{x_i}$ is a neighborhood of $x_0$ disjoint from $\bigcup_{i=1}^n W_{x_i}$, a contradiction. Hence $\bigcap \{B_x | B_x \in \mathcal{B}\} \neq \emptyset$, and for each $y \in \bigcap \{B_x | B_x \in \mathcal{B}\}$, $t = g^{-1}(y)$ is an isolated point of $K$.

Let $T$ be the set of such points $t$ in $K$. Since $T$ is open in $K$, for each $x \in A$ the set $K - (W_x \cup T)$ is compact and non-null. Then $\bigcap \{g(K - (W_x \cup T)) | x \in A\} \cap S$ is a null intersection of non-null closed subsets of a compact set, and there is a finite number of neighborhoods $W_{x_1}, \ldots, W_{x_m}$ which covers $K - (T \cup \{x_0\})$. Since $x_0$ is an accumulation point of $K$, $T$ must be infinite and hence $T \cup \{x_0\}$ has a compact infinite subset $H$ whose only accumulation point is $x_0$. Then $g(H) \cap S$ is an infinite compact subset of $S$ and must have an accumulation point $z$, a contradiction since $g^{-1}(z)$ is an isolated point of $K$.

Thus it follows that $x_0 \in f^{-1}(y_0)$ and $f^{-1}(y_0)$ is closed. By Theorem 5, $f$ is continuous.

Spaces satisfying the first axiom of countability obviously have property $K$. In conclusion there is given an example of a space in
which property $K$ holds but local countability does not.

Let $X = \cup_{n=-\infty}^{\infty} S_n$ be a planar set of points where for

$n \leq 1$, \[ S_n = \{(x, y)/0 \leq x \leq n/(n + 1), y = x(n + 1)/n\}; \]

$n \geq -1$, \[ S_n = \{(x, y)/0 \geq x \geq n/(1 - n), y = x(1 - n)/n\} \text{ and} \]

$S_0 = \{(x, y)/0 = x, 0 \leq y \leq 1\}$.

For any point $(x, y) \neq (0, 0)$, let the neighborhoods of $(x, y)$ be defined by relativization of the Euclidean topology of the plane. At $(0, 0)$, define a base as composed of sets of the form $\cup_{n=-\infty}^{\infty} E_n$, where for each integer $n$, $E_n \subset S_n$ is a half-open interval with $(0, 0)$ as the endpoint. Then the first axiom of countability is not satisfied at $(0, 0)$, but the space does have property $K$ there.

References


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