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# GENERALIZATION OF COHN-VOSSEN'S THEOREM

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In this paper Cohn-Vossen's Theorem [**1**; **2**, pp. 127–133] is extended to a characterization of similarity in $E^3$. All surfaces here concerned are assumed to be orientable, closed, convex and of class $C^3$. All homeomorphisms between surfaces are assumed to be differentiable. A scalar $C^2$ function on a surface is harmonic if it satisfies the Laplace equation

$$\Delta(\phi) = 0$$

where $\Delta$ is the second differential operator of Beltrami.

LEMMA 1. (HOPF-BOCHNER [**3**; **4**]). *The only harmonic function defined on a surface is constant.*

LEMMA 2. *Given two surfaces $S$, $\overline{S}$ and a homeomorphism $h: S \to \overline{S}$ where $h$ is conformal, the ratio of the first fundamental forms $\rho = \overline{I}/I$ satisfies*

$$\Delta(\log \rho) = 2(K - \rho\overline{K})$$

*where $K$, $\overline{K}$ are the Gaussian curvatures.*

PROOF. Since the quantities on both sides of the equation are scalars it needs only to verify in a particular system of coordinates. For $C^3$ surfaces isothermal coordinates exists locally [**5**]. By employing such coordinates the verification is straightforward.

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THEOREM. *Given two surfaces $S$, $\bar{S}$ and a homeomorphism $h: S \to \bar{S}$ which preserves $KI$. Then $h$ is a similarity.*

PROOF. Let $\phi = \log(K/\bar{K}) = (1/2) \log (|\bar{g}_{ij}| / |g_{ij}|)$ then $\phi$ is a $C^2$ function on $S$. By Lemma 2, $\phi$ is harmonic. By Lemma 1, $\phi = \text{const}$. Thus

$$K/\bar{K} = \text{const}.$$

According to Cohn-Vossen's Theorem the result of $h$ followed by a homothetic transformation of proportionality constant $(K/\bar{K})^{1/2}$ is a rigid motion. Hence $h$ is a similarity.

REMARK. The conclusion of the theorem can be written in two parts: (1) $h$ is the product of a homothetic transformation and an isometry, (2) the isometry in (1) is a congruence. The first part is valid under a rather general condition. We see that our proof can be applied to the following result.

Given two compact $C^3$ Riemannian manifold $S$, $\bar{S}$ immersed in $E^3$ and a differentiable homeomorphism $h: S \to \bar{S}$ preserving $KI$; suppose $S$, $\bar{S}$ contain no pieces of developable surfaces, then $h$ is the product of a homothetic transformation and an isometry.

Hence we see that our theorem is true for rigid closed surfaces (neither orientability nor convexity nor any restriction on genus would be assumed) immersed in $E^3$ and containing no pieces of developable surfaces. For example the analytic $T$ surfaces of Alexandrov [6] constitute a class of such surfaces.

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