ON THE UNIQUENESS OF THE PROLONGATION OF AN OPEN RIEMANN SURFACE OF FINITE GENUS[1]

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1. Let F be an open Riemann surface of finite genus g. Suppose there exists a Riemann surface W with the property that there exists a conformal mapping f of F onto a proper subregion of W. Then W or, more precisely, the pair (W, f) is called a *prolongation* of F.

In the following, we assume exclusively that W is a closed surface of genus g. The existence of such a (W, f) is well known [4]. When is such a prolongation determined uniquely? This is the problem that we shall discuss in the present paper.

Such a problem has been proposed by Nevanlinna [6]. In his case the uniqueness means the following: F is said to admit a unique prolongation if, for any (W_1, f_1) and (W_2, f_2), the mapping $f_2 \circ f_1^{-1}$ can be extended to a conformal mapping of W_1 onto W_2.

THE UNIQUENESS THEOREM. *F admits the unique prolongation in the above sense if and only if F is of O_{AD}.*[2]

It was conjectured by Nevanlinna [6] and was proved by Ahlfors and Beurling [2] for $g=0$ and by Mori [5] for $g \geqq 1$.

2. We ask here the "uniqueness" in another sense: When are all the W conformally equivalent to each other? Can the only-if-part of the Uniqueness Theorem be replaced by the following weaker form: If all the W are conformally equivalent to each other, then F is of O_{AD}? It is trivially not true for $g=0$; in general, even if $f_2 \circ f_1^{-1}$ is not extendable to a conformal mapping of W_1 onto W_2, still they may be conformally equivalent. Nevertheless we have

THEOREM 1. *Let F be an open Riemann surface of finite genus $g \geqq 1$. Then all the closed Riemann surfaces of genus g which are prolongations of F are conformally equivalent to each other if and only if F is of O_{AD}.*

Its if-part is a direct consequence of the Uniqueness Theorem. To prove the only-if-part, we shall apply the theory of Teichmüller

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[2] O_{AD} is the class of Riemann surfaces which do not carry a nonconstant analytic function with finite Dirichlet integral.

space [**8**; **1**; **3**]. For simplicity, we shall treat merely the case $g \geqq 2$; with a slight modification the reasoning is applicable also to the case $g=1$.

Let T_g be the Teichmüller space of closed Riemann surfaces of genus g (for the definition the reader is referred to, e.g., Ahlfors [**1**, p. 53]). In our previous paper [**7**], we considered the set $P(F) \subset T_g$ which corresponds to the set of all the prolongations (W, f) of F. Before proving Theorem 1, we shall prove

THEOREM 2. *Let $g \geqq 2$. If $F \in O_{AD}$ then $P(F)$ consists of a single point. If $F \notin O_{AD}$ then $P(F)$ contains an interior point.*

3. **Proof of Theorem** 2. The first part is a direct consequence of the Uniqueness Theorem.

To prove the second part suppose $F \notin O_{AD}$. Then it is not difficult to show that there exists a (W, f) such that the set $E = W - f(F)$ has positive measure (see Mori [**4**] and Ahlfors-Beurling [**2**]). We can find a complex basis $\mu_j(d\bar{z}/dz)$ $(j=1, 2, \cdots, 3g-3)$ of Beltrami differentials modulo trivial Beltrami differentials on W (for the definition see Bers [**3**]) such that $\mu_j \equiv 0$ on $W - E$. In fact, let $\phi_j dz^2$ $(j=1, 2, \cdots, 3g-3)$ be a complex basis of regular analytic quadratic differentials on W such that

$$\iint_E \frac{\phi_j \bar{\phi}_k}{L} dxdy = \delta_{jk}$$

where $L|dz|^2$ is the Poincaré metric on W; then

$$\mu_j = \begin{cases} \bar{\phi}_j/L & \text{on } E, \\ 0 & \text{on } W - E \end{cases}$$

is the desired, because the local triviality of $\sum_{j=1}^{3g-3} c_j\mu_j$ implies that $c_j \iint_E |\phi_j| dxdy = 0$ and, therefore, $c_j = 0$ $(j=1, 2, \cdots, 3g-3)$ (this method is merely a modification of the one in Bers [**3**]). Let W^μ be the Riemann surface whose conformal structure is defined by the metric $ds = |dz + \mu d\bar{z}|$ on W. As is shown by Bers [**3**], the set $U = \{W^\mu; \mu = \sum_{j=1}^{3g-3} c_j\mu_j, \sum_{j=1}^{3g-3} |c_j| < \epsilon\}$ is an open set in T_g if $\epsilon > 0$ is sufficiently small. The conformal structure of $f(F)$ in any surface in U is not changed. Hence every (W^μ, f), $W^\mu \in U$, is a prolongation of F and, therefore, $U \subset P(F)$.

4. **Proof of Theorem** 1. The Teichmüller space T_g admits the projection π in the well-known manner onto the space of conformal equivalence classes of closed Riemann surfaces of genus g.

Suppose that all the prolongations W of F are conformally equivalent to each other. Then $\pi P(F)$ consists of a single point. It is fairly easy to see that, in this case, $P(F)$ is a discrete set.[3] On the other hand we showed in [7] that $P(F)$ is a connected set. Therefore we see that $P(F)$ consists of a single point and, by Theorem 2, we conclude that F is of O_{AD}.

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[3] It is stated without proof in [8, p. 168]. It can be proved easily if the reasoning in [3] is used.