ON THE UNIQUENESS OF THE PROLONGATION OF AN OPEN RIEMANN SURFACE OF FINITE GENUS

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1. Let \( F \) be an open Riemann surface of finite genus \( g \). Suppose there exists a Riemann surface \( W \) with the property that there exists a conformal mapping \( f \) of \( F \) onto a proper subregion of \( W \). Then \( W \) or, more precisely, the pair \((W,f)\) is called a prolongation of \( F \).

In the following, we assume exclusively that \( W \) is a closed surface of genus \( g \). The existence of such a \((W,f)\) is well known \([4]\). When is such a prolongation determined uniquely? This is the problem that we shall discuss in the present paper.

Such a problem has been proposed by Nevanlinna \([6]\). In his case the uniqueness means the following: \( F \) is said to admit a unique prolongation if, for any \((W_1,f_1)\) and \((W_2,f_2)\), the mapping \( f_2 \circ f_1^{-1} \) can be extended to a conformal mapping of \( W_1 \) onto \( W_2 \).

The Uniqueness Theorem. \( F \) admits the unique prolongation in the above sense if and only if \( F \) is of \( O_{AD} \).

It was conjectured by Nevanlinna \([6]\) and was proved by Ahlfors and Beurling \([2]\) for \( g = 0 \) and by Mori \([5]\) for \( g \geq 1 \).

2. We ask here the “uniqueness” in another sense: When are all the \( W \) conformally equivalent to each other? Can the only-if-part of the Uniqueness Theorem be replaced by the following weaker form: If all the \( W \) are conformally equivalent to each other, then \( F \) is of \( O_{AD} \)? It is trivially not true for \( g = 0 \); in general, even if \( f_2 \circ f_1^{-1} \) is not extendable to a conformal mapping of \( W_1 \) onto \( W_2 \), still they may be conformally equivalent. Nevertheless we have

Theorem 1. Let \( F \) be an open Riemann surface of finite genus \( g \geq 1 \). Then all the closed Riemann surfaces of genus \( g \) which are prolongations of \( F \) are conformally equivalent to each other if and only if \( F \) is of \( O_{AD} \).

Its if-part is a direct consequence of the Uniqueness Theorem. To prove the only-if-part, we shall apply the theory of Teichmüller
space \([8; 1; 3]\). For simplicity, we shall treat merely the case \(g \geq 2\); with a slight modification the reasoning is applicable also to the case \(g = 1\).

Let \(T_g\) be the Teichmüller space of closed Riemann surfaces of genus \(g\) (for the definition the reader is referred to, e.g., Ahlfors \([1, p. 53]\)). In our previous paper \([7]\), we considered the set \(P(F) \subset T_g\) which corresponds to the set of all the prolongations \((W, f)\) of \(F\). Before proving Theorem 1, we shall prove

**Theorem 2.** Let \(g \geq 2\). If \(F \in O_{AD}\) then \(P(F)\) consists of a single point. If \(F \notin O_{AD}\) then \(P(F)\) contains an interior point.

3. **Proof of Theorem 2.** The first part is a direct consequence of the Uniqueness Theorem.

To prove the second part suppose \(F \notin O_{AD}\). Then it is not difficult to show that there exists a \((W, f)\) such that the set \(E = W - f(F)\) has positive measure (see Mori \([4]\) and Ahlfors-Beurling \([2]\)). We can find a complex basis \(\mu_j(d\tilde{z}/dz)\) \((j = 1, 2, \ldots, 3g - 3)\) of Beltrami differentials modulo trivial Beltrami differentials on \(W\) (for the definition see Bers \([3]\)) such that \(\mu_j = 0\) on \(W - E\). In fact, let \(\phi_j d\tilde{z}^2\) \((j = 1, 2, \ldots, 3g - 3)\) be a complex basis of regular analytic quadratic differentials on \(W\) such that

\[
\int \int_E \frac{\phi_j\phi_k}{L} \, dxdy = \delta_{jk}
\]

where \(L|dz|^2\) is the Poincaré metric on \(W\); then

\[
\mu_j = \begin{cases} 
\frac{\tilde{\phi}_j}{L} & \text{on } E, \\
0 & \text{on } W - E
\end{cases}
\]

is the desired, because the local triviality of \(\sum_{j=1}^{3g-3} c_j \mu_j\) implies that \(c_j \int_E |\phi_j| \, dxdy = 0\) and, therefore, \(c_j = 0\) \((j = 1, 2, \ldots, 3g - 3)\) (this method is merely a modification of the one in Bers \([3]\)). Let \(W^\mu\) be the Riemann surface whose conformal structure is defined by the metric \(ds = |dz + \mu d\tilde{z}|\) on \(W\). As is shown by Bers \([3]\), the set \(U = \{ W^\mu; \mu = \sum_{j=1}^{3g-3} c_j \mu_j, \sum_{j=1}^{3g-3} |c_j| < \epsilon \}\) is an open set in \(T_g\) if \(\epsilon > 0\) is sufficiently small. The conformal structure of \(f(F)\) in any surface in \(U\) is not changed. Hence every \((W^\mu, f), W^\mu \in U\), is a prolongation of \(F\) and, therefore, \(U \subset P(F)\).

4. **Proof of Theorem 1.** The Teichmüller space \(T_g\) admits the projection \(\pi\) in the well-known manner onto the space of conformal equivalence classes of closed Riemann surfaces of genus \(g\).
Suppose that all the prolongations $W$ of $F$ are conformally equivalent to each other. Then $\pi P(F)$ consists of a single point. It is fairly easy to see that, in this case, $P(F)$ is a discrete set.\footnote{It is stated without proof in [8, p. 168]. It can be proved easily if the reasoning in [3] is used.} On the other hand we showed in [7] that $P(F)$ is a connected set. Therefore we see that $P(F)$ consists of a single point and, by Theorem 2, we conclude that $F$ is of $O_{A_D}$.

**References**


