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SOME RESULTS ON TAME DISKS AND SPHERES IN E^3

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It is well known that there are cells and spheres of dimensions one and two in Euclidean 3-space E^3 which fail to be locally tame at a single point. Artin and Fox [1] have given many examples. Several authors have given characterizations, such as [**11**; **12** and **13**], of the tame cells among the "almost polyhedral" cells. The results in this paper are of the same general nature.

We use the term "disk" to mean a topological closed 2-cell. If D is a disk, then D^0 denotes the interior of D and BdD denotes the boundary of D. The same notation also will be applied to an arc (a topological closed 1-cell).

The Theorem 0 below is stated without proof here, since a complete proof appears in Lemma 5.1 of [**7**]. We have used this result extensively and have noticed many modifications of it in the literature. It therefore seems useful to state it explicitly.

THEOREM 0. *Let D be a disk in E^3 which is locally polyhedral except at points of BdD. If U is any open set containing D^0, then there is a 3-cell C^3 in $U \cup BdD$ such that* (i) $D \subset C^3$, (ii) *D spans BdC^3,* (iii) *and C^3 is locally polyhedral except on BdD.*

THEOREM 1. *Let D be a disk in E^3 and let A be a tame arc on D spanning BdD. If D is locally polyhedral except on A, then D is tame.*

PROOF. A remark will suffice to show that D is locally tame at each point of D^0. In [**4**] it is shown that each point x in A^0 lies in the

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interior of a disk P_x in D^0 and that P_x is the union of two tame disks meeting in an arc on the boundary of each. Then, by Theorem 1 of [5], P_x is tame and hence D is locally tame at x.

In view of Theorem 2 of [10] we may assume that D is locally polyhedral except on a straight line interval A. At an endpoint p of A we attach an interval B in the same line as A. There is a subdisk D' of D such that A spans BdD' and $D' \cap B = p$. Furthermore, D' is locally tame except perhaps on BdA. By Theorem 0 and [3] there is a 2-sphere S which is locally tame except on BdA such that $S \cap D = BdD'$, $S \cap B = p$ and the components of $(A \cup B) - p$ lie in different components of $E^3 - S$. But then S is pierced at the point p by the interval $A \cup B$. Hence, by Theorem 1 of [11], S is locally tame at p. Now on the sphere S we may select a tame arc A_1 such that $A_1 \cap D = p$. Let U be an open set in E^3 such that U contains D^0 but fails to meet $A_1 \cup BdD$. In view of Theorem 0 and [3], there is a sphere S_1 in $U \cup BdD$ such that D^0 lies in the bounded component of $E^3 - S_1$ and S_1 is locally tame except on BdA. Since the tame arc $A \cup A_1$ pierces S_1 at p, S_1 is locally tame at p. It then follows that BdD is locally tame at p. Thus D is locally tame everywhere and is therefore tame by [2] and [12].

THEOREM 2. *Let S be a topological 2-sphere in E^3 and let A be a tame arc on S. If S is locally polyhedral except on A, then S is tame.*

PROOF. On S we select a disk D having A as a spanning arc of BdD while BdD is locally polygonal except at BdA. By Theorem 1, D is tame. Then by [12] or [13], $S - D^0$ is tame and Theorem 9.3 of [13] implies that S is tame.

Combining Theorem 2 of [5] with Theorem 2 above, the following result is immediate.

COROLLARY 1. *Let S be a topological 2-sphere in E^3 and let G be a tame graph (finite, connected 1-complex) on S. If S is locally polyhedral (or is locally tame) except on G, then S is tame.*

THEOREM 3. *Let D be a disk in E^3 and let A be a tame arc on D. If D is locally polyhedral except on A, then D is tame.*

PROOF. That D is locally tame at each point of D^0 can be established in the same manner as was used in Theorem 1. In view of Theorem 9 of [2] we can throw D by a space homeomorphism h onto a disk D' which is locally polyhedral except on BdD'. Theorem 0 implies that D' lies on a sphere S which is locally polyhedral except on BdD'. Then S is locally tame except perhaps on $h(A)$. But by Theorem 2, S is tame and hence D' and D are tame.

In [5] a characterization of tame graphs is given. The above results permit another such characterization which we state without proof.

COROLLARY 2. *Let G be a graph in E^3 such that each 1-cell in G is tame. If the star of every vertex of G lies on a disk, then G is tame.*

The construction used in the proof of Theorem 1 has suggested the following definition: Let S be a topological 2-sphere in E^3 and let T be a tame arc in E^3 such that $S \cap T$ is an endpoint of T. Then T is *unknotted relative to* S if T lies on the boundary of a disk D in E^3 such that $D \cap S$ is an arc in BdD. If no such disk exists, then T is *knotted relative to* S. (This definition may be compared with similar properties used by Harrold [9].)

THEOREM 4. *Let S be a topological 2-sphere in E^3 which is locally polyhedral except at a point p. If there is a tame arc T in $(E^3 - S) \cup p$ with endpoint p and if T is unknotted relative to S, then S is tame.*

PROOF. By assumption there is a disk D containing T in BdD such that $S \cap D = S \cap BdD$ is an arc A. Using the Bing approximation theorem [3] replace D by a disk D' which is locally polyhedral except on $A \cup T$. Then since S is locally polyhedral except at p, we may take A to be locally polygonal except at p. Then D' is tame by Theorem 3 and in particular the arc A is tame. So S is tame by Theorem 2.

This last result is of interest when considering such examples as 3.1 of [1]. Also in this connection, we note that any sphere in E^3 is accessible by intervals at a dense subset from each of its complementary domains.

The following result seems to have escaped notice in previous discussions of almost polyhedral spheres.

THEOREM 5. *Let S be a 2-sphere in E^3 which is locally polyhedral except at a point p and let A_1 and A_2 be two arcs on S. If $A_1 \cap A_2 = p$ and p is an endpoint of both arcs, then A_1 and A_2 are equivalently imbedded in E^3; that is, there is a space homeomorphism carrying A_1 onto A_2.*

PROOF. Clearly A_1 and A_2 lie on the boundary of a disk D in S. We join the distinct endpoints of A_1 and A_2 by a polygonal arc J_0, disjoint from A_1 and A_2 except at the endpoints, to form the disk D. On D we select a sequence of disjoint polygonal arcs $\{J_i\}$ converging to the point p and such that each J_i has its endpoints on A_1 and A_2. We enclose each J_i, $i = 0, 1, 2, \cdots$, in a polyhedral 2-sphere S_i meeting S in a simple closed curve which contains J_i in its interior.

The spheres S_i are taken to be pairwise disjoint. Then there is a space homeomorphism h_i which is the identity except inside S_i and carries the endpoint of J_i on A_1 onto the endpoint of J_i on A_2 while leaving the sphere S invariant. This homeomorphism h_i is so chosen that $h_i(A_1)\cap A_2$ is a single point. Applying these mappings h_i sequentially, the limit mapping h is obviously a space homeomorphism such that $h(A_1)\cap A_2$ is a sequence of points $\{p_i\}$ converging to p. This gives us a sequence of 2-cells D_i on S bounded by A_2 and $h(A_1)$ such that $D_i\cap D_j$ is empty if $j\neq i-1,\ i,\ i+1$ and otherwise $D_i\cap D_{i+1}=p_{i+1}$. Since these cells D_i are all tame we can construct 2-spheres S_i' such that $D_i-p_i-p_{i+1}$ lies in the bounded component of E^3-S_i' and the S_i' intersect as do the D_i. Then space homeomorphisms h_i' which are the identity except inside S_i' will move $h(A_1)$ entirely onto A_2 while leaving S invariant and the point p fixed. Therefore A_1 and A_2 are equivalently imbedded in E^3.

COROLLARY 3. *If A_1 and A_2 are arcs passing through p in the 2-sphere S of Theorem 5 and if $A_1\cap A_2=p$, then A_1 and A_2 are equivalently imbedded in E^3.*

In [8] Harrold and Moise show that each 2-sphere S in S^3 such that S is locally polyhedral except at a point p has a closed 3-cell as the closure of one of its complementary domains. It is clear from the above observations that any almost polyhedral spanning disk in this 3-cell which has the point p on its boundary will contain arcs imbedded equivalently to those on S itself. But in addition we have the following result.

COROLLARY 4. *Every 2-sphere in E^3, locally polyhedral except at one point, is accessible from both of its complementary domains by arcs imbedded in E^3 equivalently to arcs on the sphere.*

We conclude by collecting several results which are very easily established but which seem to be of interest.

THEOREM 6. *Let D be a disk (S be a sphere) in E^3 which is locally polyhedral except at a point p in D^0 (at a point p) and suppose all arcs on $D(S)$ are tame. Then $D(S)$ is tame.*

PROOF. By [6], $D(S)$ is pierced by a tame arc at p. Then Theorem 1 of [11] may be applied to prove that $D(S)$ is tame.

COROLLARY 5. *If S is a 2-sphere in E^3 on which every arc is tame and if S is locally polyhedral except on a closed, totally disconnected set M, then S is tame.*

PROOF. Pass an arc A on S through M. A is tame and Theorem 2 applies.

COROLLARY 6. *Let D be a disk (S be a sphere) in E^3 which is locally polyhedral except at a point p of D^0 (of S) and suppose $D(S)$ is wild at p. Then every arc through p is wild.*

PROOF. That some such arc through p is wild follows from Theorem 6 and then the conclusion follows from Theorem 5.

We remark that Corollary 6 may be extended in the obvious way to a sphere which is wild at finitely many points. Furthermore, it follows that a disk cannot be locally wild at an isolated point without containing a wild arc. Stating this conversely, a disk on which all arcs are tame cannot be locally wild at an isolated point (Corollary 6) and hence cannot be wild only on a closed totally disconnected set of points (Corollary 5).

This last remark is of interest in view of the recent announcement by R. H. Bing of a wild sphere on which all arcs are tame.

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