TOPOLOGICAL AND MEASURE THEORETIC PROPERTIES OF ANALYTIC SETS

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1. **Introduction.** G. Choquet's definition [1] of an analytic set in a topological space as the continuous image of a $K_{\sigma\delta}$ has proved to be a very fruitful one. In this paper we study some properties of analytic sets which are not necessarily included in a K_σ. In particular, we prove that an analytic set is Lindelöf, capacitable, measurable and can be approximated in measure by compact sets from within for large classes of capacities and measures.

2. **Topological properties of analytic sets.**

2.1. DEFINITIONS. Our definition of an analytic set below is a slight modification of Choquet's in that we do not impose any conditions on X.

1. ω denotes the set of all non-negative integers.
2. $K(X)$ denotes the family of compact sets in X.
3. $\mathfrak{F}(X)$ denotes the family of closed sets in X.
4. A is analytic iff A is the continuous image of a $K_{\sigma\delta}(X)$ for some X.
5. A is Souslin H iff

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} E(s_0, \cdots, s_n)$$

where S is the set of all sequences of non-negative integers and $E(s_0, \cdots, s_n) \in H$ for every $s \in S$ and $n \in \omega$.

6. A is Lindelöf iff every open covering of A has a countable subcovering of A.
7. $\bar{A}$ denotes the closure of A.

2.2. LEMMA. *Let f be a continuous function on D to a topological space X and, for every $n \in \omega$, let A_n be compact, $A_{n+1} \subset A_n$, $\bigcap_{n\in\omega} A_n \subset D$, $B_n = f(D \cap A_n)$. Then*

(i) *if U is open in X and $f(\bigcap_{n\in\omega} A_n) \subset U$ then, for some $n \in \omega$, $B_n \subset U$,*

(ii) *if X is Hausdorff then*

$$f\left(\bigcap_{n\in\omega} A_n\right) = \bigcap_{n\in\omega} B_n = \bigcap_{n\in\omega} \bar{B}_n.$$

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PROOF. Let $C'=\bigcap_{n\in\omega} A_n$ and $C=f(C')$. Suppose U is open in X and $C\subset U$. Let $U'=f^{-1}(U)$. Then U' is open in D and $C'\subset U'$. Hence, for some $n\in\omega$, $D\cap A_n\subset U'$ and

$$B_n = f(D\cap A_n)\subset f(U') = U.$$

Next, suppose X is Hausdorff and $y\notin C$. Since C' is compact so is C. Thus, there exists an open set U such that $C\subset U$ and $y\notin \overline{U}$. Then, by part (i), for some $n\in\omega$, $\overline{B}_n\subset \overline{U}$ and $y\notin \overline{B}_n$. Thus,

$$\bigcap_{n\in\omega} \overline{B}_n \subset C.$$

On the other hand, we always have

$$C\subset \bigcap_{n\in\omega} B_n \subset \bigcap_{n\in\omega} \overline{B}_n.$$

2.3. THEOREM. *If E is analytic then E is Lindelöf.*

PROOF. Since E is the continuous image of a $K_{\sigma\delta}(X)$, for some X, let

$$D = \bigcap_{i\in\omega}\ \bigcup_{j\in\omega} d(i,j),$$

where the $d(i,j)$ are compact, f be continuous on D and $E=f(D)$. Suppose F is an open covering of E that has no countable subcovering of E. Then, by recursion, we define a sequence k with $k_i\in\omega$ such that for any $n\in\omega$, no countable subfamily of F covers

$$f\left(D\cap \bigcap_{i=0}^{n} d(i,k_i)\right).$$

To this end, we observe that, since

$$E = \bigcup_{j\in\omega} f(D\cap d(0,j)),$$

there exists $k_0\in\omega$ such that no countable subfamily of F covers $f(D\cap d(0,k_0))$. Having defined k_i for $i=0,\cdots,n$, since

$$f\left(D\cap \bigcap_{i=0}^{n} d(i,k_i)\right) = \bigcup_{j\in\omega} f\left(D\cap \bigcap_{i=0}^{n} d(i,k_i)\cap d(n+1,j)\right)$$

we see that there exists $k_{n+1}\in\omega$ such that no countable subfamily of F covers

$$f\left(D\cap \bigcap_{i=0}^{n+1} d(i,k_i)\right).$$

Let

$$A_n = \bigcap_{i=0}^{n} d(i, k_i),$$

$$B_n = f(D \cap A_n),$$

$$C = f\left(\bigcap_{n\in\omega} A_n\right).$$

Since

$$\bigcap_{n\in\omega} A_n = \bigcap_{i\in\omega} d(i, k_i) \subset D$$

and C is compact, let U be the union of a finite subfamily of F such that $C \subset U$. Then, by Lemma 2.2, for some $n \in \omega$, $B_n \subset U$ contradicting the fact that B_n could not be covered by a countable subfamily of F.

2.4. Corollary. *If E is analytic in a metric space then E is separable.*

2.5. Theorem. *If E is analytic in a Hausdorff X then E is Souslin $\mathfrak{F}(X)$.*

Proof. Let $E = f(D)$ where f is continuous on D,

$$D = \bigcap_{i\in\omega} \bigcup_{j\in\omega} d(i, j),$$

with $d(i, j)$ compact. For every $(n+1)$-tuple $(k_0, \cdots, k_n)$ with $k_i \in \omega$, let

$$B(k_0, \cdots, k_n) = f\left(D \cap \bigcap_{i=0}^{n} d(i, k_i)\right)$$

so that $\overline{B}(k_0, \cdots, k_n) \in \mathfrak{F}(X)$. If S denotes the set of all sequences k with $k_i \in \omega$, we shall show that

$$E = \bigcup_{k\in S} \bigcap_{n\in\omega} \overline{B}(k_0, \cdots, k_n).$$

Indeed, since

$$D = \bigcup_{k\in S} \bigcap_{i\in\omega} d(i, k_i)$$

we have

$$E = \bigcup_{k\in S} f\left(\bigcap_{i\in\omega} d(i, k_i)\right)$$

and for every $k \in S$

$$f\left(\bigcap_{i\in\omega} d(i, k_i)\right) = f\left(\bigcap_{n\in\omega} \bigcap_{i=0}^{n} d(i, k_i)\right) = \text{(by 2.2)}$$

$$= \bigcap_{n\in\omega} f\left(D \cap \bigcap_{i=0}^{n} d(i, k_i)\right) = \bigcap_{n\in\omega} \overline{B}(k_0, \cdots, k_n).$$

2.6. COROLLARY. *If E is a metric space and analytic then E can be imbedded in a complete separable metric space X and, for every such X, E is Souslin $\mathfrak{F}(X)$. Thus, in this case, E is an absolute analytic set in the classical sense and hence it is the continuous image of the set of irrationals.* (*See* [3] *for results on analytic sets in separable metric spaces.*)

2.7. REMARK. Theorem 2.5 was first proved by the author in [5]. The proof we give here is shorter and simpler. Corollary 2.6 was first obtained by G. Choquet [2].

3. **Capacitability of analytic sets.** We consider here only special kinds of capacities. For more general definitions and results about capacities see G. Choquet [1].

3.1. DEFINITIONS. 1. ϕ is a *capacity* on X iff ϕ is a function on $K(X)$ to the reals (including $\pm\infty$) such that: if A and B are compact and $A \subset B$ then $\phi(A) \leqq \phi(B)$; if $\phi(A) < a$ then there exists an open set U such that $A \subset U$ and, for every compact $A' \subset U$, $\phi(A') < a$.

2. If ϕ is a capacity on X then for any $A \subset X$ we let

$$\phi_*(A) = \sup\{t : t = \phi(C) \text{ for some compact } C \subset A\}$$

and

$$\phi^*(A) = \inf\{t : t = \phi_*(U) \text{ for some open } U \supset A\}.$$

3. If ϕ is a capacity on X then A is ϕ-capacitable iff

$$\phi_*(A) = \phi^*(A).$$

4. If ϕ is a capacity on X and $E \subset X$ then ϕ is of order (1a) on E iff, for every sequence A with $A_i \subset A_{i+1} \subset E$, we have

$$\phi^*\left(\bigcup_{i\in\omega} A_i\right) = \lim_i \phi^*(A_i).$$

(This is a slight extension of Choquet's definition of a capacity alternating of order $\mathfrak{A}_{1,a}$.)

3.2. THEOREM. *If E is analytic in X, ϕ is a capacity on X of order* (1a) *on E then E is ϕ-capacitable.*

PROOF. Let $E=f(D)$ where f is continuous on D,

$$D = \bigcap_{i\in\omega} \bigcup_{j\in\omega} d(i, j),$$

the $d(i, j)$ are compact, and $d(i, j)\subset d(i, j+1)$ for every $i\in\omega$, $j\in\omega$. In order to see that $\phi_*(E)=\phi^*(E)$, we let $a<\phi^*(E)$ and show that, for some compact $C\subset E$, $\phi(C)\geqq a$. By recursion, we define a sequence k with $k_i\in\omega$ as follows. Since

$$E = \bigcup_{j\in\omega} f(D\cap d(0, j))$$

let $k_0\in\omega$ be such that

$$a < \phi^*(f(D\cap d(0, k_0))).$$

Having $k_0, \cdots, k_n$ so that

$$a < \phi^*\left(f\left(D\cap \bigcap_{i\in 0}^{n} d(i, k_i)\right)\right),$$

since

$$f\left(D\cap \bigcap_{i=0}^{n} d(i, k_i)\right) = \bigcup_{j\in\omega} f\left(D\cap \bigcap_{i=0}^{n} d(i, k_i)\cap d(n+1, j)\right),$$

let k_{n+1} be such that

$$a < \phi^*\left(D\cap \bigcap_{i=0}^{n+1} d(i, k_i)\right).$$

Set

$$A_n = \bigcap_{i=0}^{n} d(i, k_i),$$

$$C = f\left(\bigcap_{n\in\omega} A_n\right).$$

Then, since

$$\bigcap_{n\in\omega} A_n = \bigcap_{i\in\omega} d(i, k_i) \subset D,$$

we see that C is compact, $C\subset E$ and, by 2.2, for any open set $U\supset C$ there exists $n\in\omega$ such that $f(D\cap A_n)\subset U$. Hence

$$a < \phi^*(D\cap A_n) \leqq \phi^*(U).$$

Thus, $a \leqq \phi^* C$ and, since compact sets are ϕ-capacitable, we have $\phi(C) = \phi^*(C) \geqq a$.

4. Measurability of analytic sets.

4.1. DEFINITIONS. 1. μ is a Carathéodory measure on X iff μ is a function on the family of all the subsets of X to the non-negative reals (including ∞) such that $\mu(0) = 0$ and if

$$A \subset \bigcup_{n \in \omega} B_n \subset X$$

then

$$\mu(A) \leqq \sum_{n \in \omega} \mu(B_n).$$

2. If μ is a Carathéodory measure on X and $A \subset X$ then A is μ-measurable iff for every $T \subset X$

$$\mu(T) = \mu(T \cap A) + \mu(T - A).$$

3. $M(X, E)$ will denote the set of all Carathéodory measures on X such that:

(i) for every $A \subset E$ there is a μ-measurable A' with $A \subset A'$ and $\mu(A) = \mu(A')$.

(ii) all compact subsets of E are μ-measurable.

(iii) if A is compact, $A \subset E$, $T \subset X$, $\epsilon > 0$ then there exists an open set U such that $A \subset U$ and $\mu(T \cap U) \leqq \mu(T \cap A) + \epsilon$.

4.2. THEOREM. *If E is analytic in X, $\mu \in M(X, E)$, $a < \mu(E)$ then there exists a compact C such that $C \subset E$ and $\mu(C) > a$.*

PROOF. For any sequence A such that $A_i \subset A_{i+1} \subset E$ we have

$$\mu\left(\bigcup_{i \in \omega} A_i\right) = \lim_i \mu(A_i).$$

Let $E = f(D)$ where f is continuous on D,

$$D = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j),$$

the $d(i, j)$ are compact and $d(i, j) \subset d(i, j+1)$. As in the proof of 3.2, by recursion we can define a sequence k with $k_i \in \omega$ such that, for any $n \in \omega$:

$$a < \mu\left(f\left(D \cap \bigcap_{i=0}^{n} d(i, k_i)\right)\right).$$

Let

$$A_n = \bigcap_{i=0}^{n} d(i, k_i),$$

$$C = f\left(\bigcap_{n\in\omega} A_n\right)$$

then C is compact, $C\subset E$ and, by 2.2, for any open set $U\supset C$ there exists $n\in\omega$ such that $f(D\cap A_n)\subset U$ and therefore

$$a < \mu(f(D \cap A_n)) \leqq \mu(U).$$

Thus, $\mu(C)\geqq a$.

4.3. THEOREM. *If E is analytic in X, $\mu\in M(X, E)$ then E is μ-measurable.*

PROOF. Let $T\subset X$ and $\mu(T)<\infty$. We define a $n\in\omega$ measure ν on X by letting $\nu(A)=\mu(T\cap A)$ for every $A\subset X$.

Clearly $\nu\in M(X, E)$ and $\nu(E)<\infty$. Then, by 4.2, there exists a $C\subset E$ such that C is a $K_\sigma(X)$ and is ν-measurable and $\nu(C)=\nu(E)$, so that $\nu(E-C)=0$. Thus, $E-C$ is ν-measurable and hence E is ν-measurable and

$$\mu(T) = \nu(T) = \nu(T \cap E) + \nu(T - E) = \mu(T \cap E) + \mu(T - E).$$

4.4. COROLLARY. *If X is a complete separable metric space, E is closed in X, and $\mu\in M(X, E)$ then E is μ-measurable.*

4.5. REMARK. If X is Hausdorff, E is analytic in X, μ is a Carathéodory measure on X such that closed sets are μ-measurable, then E is μ-measurable and

$$\mu(E) = \sup_{C\in K(X);C\subset E} \mu(C)$$

since in this case E is Souslin $\mathfrak{F}(X)$. (See e.g. Saks [4].)

4.6. REMARK. In the proof of Theorem 4.2 we do not use all the properties of a measure μ in $M(X, E)$. Part (iii) in the definition of $M(X, E)$ may be replaced by

(iii)′ if A is compact, $A\subset E$, $\epsilon>0$ then there exists an open set U such that $A\subset U$ and $\mu(U)\leqq\mu(A)+\epsilon$ without affecting the validity of 4.2.

For the proof of 4.3 however (iii)′ is not enough, but we may replace (iii) by

(iii)″ if A is compact, $A\subset E$, $\epsilon>0$, $T\subset X$ and $\mu(T)<\infty$ then there exists an open set U such that $A\subset U$ and $\mu(T\cap U)\leqq\mu(T\cap A)+\epsilon$.

Part (iii) will be satisfied if we have (iii)′ and, for every compact $A \subset E$, $\mu(A) < \infty$, or if we have

(iii)‴ if A is compact, $A \subset E$, $\epsilon > 0$ then there exists an open set U such that $A \subset U$ and $\mu(U - A) < \epsilon$.

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